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Path integration on hyperbolic spaces

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Abstract. Quantum mechanics on the hyperbolic spaces of rank one is discussed using a path integration technique. Hyperbolic spaces are multi-dimensional generalizations of the hyperbolic plane, i.e. the Poincaré upper half-plane endowed with a hyperbolic geometry. We evaluate the path integral on $S_1 \cong \mathrm{SO}(n,1)/\mathrm{SO}(n)$ and $S_2 \cong \mathrm{SU}(n,1)/\mathrm{S}[\mathrm{U}(1)\times\mathrm{U}(n)]$ in a particular coordinate system, yielding explicitly the wavefunctions and the energy spectrum. Furthermore we can exploit a general property of all these spaces, namely that they can be parameterized by a pseudopolar coordinate system. This allows the path integration to be separated into one over spheres and an additional path integration over the remaining hyperbolic coordinate, effectively yielding a path integral for a modified Pöschl–Teller potential. Only continuous spectra can exist in all cases. For all the hyperbolic spaces of rank one we find a general *formula* for the largest lower bound (zero-point energy) of the spectrum which is given by $E_0 = (\hbar^2/8m)(m_\alpha + 2m_{2\alpha})^2$ (m_α and $m_{2\alpha}$ denote the dimensions of the root subspaces corresponding to the roots α and 2α, respectively). The case, where a constant magnetic field on S_1 is incorporated, is also discussed.

1. Introduction

The study of hyperbolic space has a long history, starting with the pioneering work of Fricke and Klein [1] and Poincaré [2]. The remarkable property of spaces with constant negative curvature (Gaussian curvature $K=-1$, Riemann curvature $R=-2$), in particular the hyperbolic plane $\mathcal{H}$ (Poincaré upper half-plane, Lobaschevsky plane, Poincaré disc) is that under a group action (Fuchsian group) a tesselation of the entire space can be achieved, the actual tesselation consisting of arbitrary hyperbolic polyhedrals with geodesics (geodesic planes) as boundaries. The specific feature of these geometries enables one to have finite polyhedrals which may be compact as well as non-compact. One of the important mathematical properties of the hyperbolic polygons tesselating the hyperbolic plane is that they can be identified with Riemann surfaces of a particular genus.

However, the actual study of Laplacians on these polyhedrals in general, and on Riemann surfaces in particular, respectively, turns out to be extremely difficult, whereas the Laplacians for the free motion (often a constant magnetic field can also be included, so-called Maass–Laplacians) on the entire spaces are relatively easy to solve, including the determination of the (free) wavefunctions, the (continuous) spectra and the energy-dependent Green functions (resolvent kernels). One of the

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original hopes in the study of trace formulae was that they could provide some information about these spectra. This was one of Selberg's motives [3, 4] for developing his famous trace formula for $\mathrm{PSL}(2,\mathbb{R})$.

Only recently have physicists have become interested in trace formulae, in particular for Riemann surfaces. This interest emerged from three reasons:

(i) In bosonic string theory one deals with a path integral formulation due to Polyakov [5, 6], where in the subsequent integration (see [7] and references therein) over metrics one has to sum over all genera of the (closed) surfaces the world-sheet can take on, in fact a perturbation expansion, and integrate over all possible deformations of these surfaces for a fixed genus which is an integration over the Teichmüller space. The closed surfaces can be identified with polygons tesselating the hyperbolic plane, cf the uniformization theorem for Riemann surfaces.

(ii) In the semiclassical regime, trace formulae emerge for classically chaotic systems in the context of periodic orbit theory which was systematically developed by Gutzwiller [8], and it did not take long before he rediscovered [9] the Selberg trace formula, albeit in a different context, and that furthermore, the Selberg trace formula is an *exact* formula.

(iii) In quantum chaos the study of classical and quantum motion on these Riemann surfaces emerges quite naturally [10] because the classical motion is highly chaotic and two-dimensional systems are the simplest systems where this can occur. By a thorough study on the lowest genus $g = 2$ case Aurich *et al* [11] achieved much in understanding the classical and quantum properties (quantum chaology) of this particular system.

The simplest case of the hyperbolic plane is easy to generalize to higher dimensions, i.e. hyperbolic space. For example, the metric on the Poincaré upper half-plane $\mathcal{H} = \{z = x + \mathrm{i}y | x \in \mathbb{R}, y > 0\}$ endowed with the hyperbolic metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2} \tag{1.1}$$

is generalized to (e.g. [12–20])

$$\mathrm{d}s^2 = \frac{\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \cdots + \mathrm{d}x_{n-1}^2 + \mathrm{d}y^2}{y^2} \tag{1.2}$$

with the hyperbolic space $\mathcal{H}^n \cong \mathrm{SO}(n,1)/\mathrm{SO}(n)$ defined as $\mathcal{H}^n = \{(x_1,\ldots,x_{n-1},y) | x_1,\ldots,x_{n-1} \in \mathbb{R}, y > 0\}$. Of course, we have $\mathcal{H} = \mathcal{H}^2 \cong \mathrm{SO}(2,1)/\mathrm{SO}(2)$. The space $\mathcal{H}^n$ will also be referred to as S_1.

Generally these hyperbolic spaces have a common structure. One considers the Hermitean $p + q$ form (see e.g. [21–23])

$$Q_e^{(p,q)} = y_1^* x_1 + \ldots y_p^* x_p - y_{p+1}^* x_{p+1} - \ldots - y_{p+q}^* x_{p+q} = e \tag{1.3}$$

and asks for the Lie group of linear operators acting on F^{p+q} which leaves it invariant. Here F can be $F = \mathbb{R}$, $F = \mathbb{C}$ or $F = H$, respectively, where H denotes the field of quaternions. Of course, $x, y \in F^{(p+q)}$. For the hyperboloids leading to the study of hyperbolic spaces, we have $p = n$, $q = 1$ and $e = -1$, say.

In this paper we are going to study the path integral formulation on the four multi-dimensional hyperbolic spaces of rank one, namely [16] $S_1 \equiv \mathcal{H}^n \cong$

$\mathrm{SO}(n,1)/\mathrm{SO}(n)$, without and with magnetic field, $S_2 \cong \mathrm{SU}(n,1)/\mathrm{S}[\mathrm{U}(1) \times \mathrm{U}(n)]$ and $S_3 \cong \mathrm{Sp}(n,1)/[\mathrm{Sp}(1) \times \mathrm{Sp}(n)]$, respectively. We can also include in our discussion the remaining rank one case, the so-called exceptional space $S_4 \cong F_{4(-20)}/\mathrm{Spin}(9)$ [21, 24]. The group pairs $(\mathrm{SO}(n,1), \mathrm{SO}(n))$, $(\mathrm{SU}(1,n), S[U(1) \times U(n)])$, $(\mathrm{Sp}(n,1), [\mathrm{Sp}(1) \times \mathrm{Sp}(n)])$ and $(F_{4(-20)}, \mathrm{Spin}(9))$ form so-called Gelfand pairs, e.g. [21]. (Note that the convention whether one uses $\mathrm{SO}(n,1)$ or $\mathrm{SO}(1,n)$ etc, respectively, depends on the signature of the metric one uses in $Q_e^{(p,q)}$, say.)

We evaluate the wavefunctions and energy spectrum explicitly. Furthermore an analytic expression for the Green functions (resolvent kernel) will be given in the case of S_1.

The rest of this paper is organized as follows:

In the next section, a summary of an appropriate path integral formulation on curved spaces is given, and the relevant formulae for time transformation and separation of variables in path integrals, respectively, are cited.

In the following two sections the path integral treatments of the spaces S_1 and S_2, respectively, will be given. We evaluate the path integral in the rectangular coordinate formulation, generalizing the Poincaré upper half-plane. For S_1 a path integral treatment in pseudospherical polar coordinate system will also be given, thus exploiting the underlying $\mathrm{SO}(n)$ symmetry. For S_2 a separation in terms of $\mathrm{SU}(n-1)$ polar coordinates is possible and will be presented as well.

In the fifth section we use some general results from harmonic analysis on hyperbolic spaces of rank one to achieve a complete separation in terms of a path integration on spheres and a hyperbolic coordinate. The remaining path integral over the hyperbolic coordinate turns out to be the path integral for the modified Pöschl–Teller potential which can be explicitly solved.

As we shall see many recent path integral calculations are needed in order to evaluate the present ones, demonstrating the power of the whole technique. The sixth section contains a discussion and a summary.

In appendix A we briefly review, for clarity, how to do the path integration on the covering unit sphere of $\mathrm{SU}(n,1)$, and in appendix B we point out some relationships between the path integration of the covering unit spheres of the group spaces of the group manifolds $\mathrm{SO}(n,1)$, $\mathrm{SU}(n,1)$ and $\mathrm{Sp}(n,1)$, respectively.

2. Formulation of path integrals, time transformations, and separation of variables

In order to set up our notation we proceed in the canonical way for path integrals on curved spaces [25–28]. We start by considering the generic case where the classical Lagrangian corresponding to the line element $\mathrm{d}s^2 = g_{ab}\mathrm{d}q^a\mathrm{d}q^b$ of the classical motion in some D-dimensional Riemannian space is given by

$$\mathcal{L}_{\mathrm{Cl}}(q,\dot{q}) = \frac{m}{2}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 - V(q) = \frac{m}{2}g_{ab}\dot{q}^a\dot{q}^b - V(q). \tag{2.1}$$

The quantum Hamiltonian is *constructed* by means of the Laplace–Beltrami operator Δ_{LB}

$$H = -\frac{\hbar^2}{2m}\Delta_{\mathrm{LB}} + V(q) = -\frac{\hbar^2}{2m}\frac{1}{\sqrt{g}}\frac{\partial}{\partial q_a}g^{ab}\sqrt{g}\frac{\partial}{\partial q_b} + V(q) \tag{2.2}$$

as a *definition* of the quantum theory on a curved space. Here $g = \det(g_{ab})$ and $(g^{ab}) = (g_{ab})^{-1}$. The scalar product for wavefunctions on the manifold reads

$$(f, g) = \int \mathrm{d}q \sqrt{g}\, f^*(q) g(q) \tag{2.3}$$

and the momentum operators which are Hermitean with respect to this scalar product are given by

$$p_a = \frac{\hbar}{\mathrm{i}} \left(\frac{\partial}{\partial q^a} + \frac{\Gamma_a}{2} \right) \qquad \Gamma_a = \frac{\partial \ln \sqrt{g}}{\partial q^a}. \tag{2.4}$$

We now rewrite the metric tensor as a product according to $g_{ab} = h_{ac} h_{cb}$ [29]. Then we obtain for the Hamiltonian (2.2)

$$H = -\frac{\hbar^2}{2m} \Delta_{\mathrm{LB}} + V(q) = \frac{1}{2m} h^{ac} p_a p_b h^{cb} + \Delta V(q) + V(q) \tag{2.5}$$

and for the path integral

$$\begin{aligned} K(q'', q'; T) & \\ &= \int\limits_{q(t')=q'}^{q(t'')=q''} \sqrt{g}\, \mathcal{D}q(t) \exp\left\{ \frac{\mathrm{i}}{\hbar} \int_{t'}^{t''} \left[\frac{m}{2} h_{ac} h_{cb} \dot{q}^a \dot{q}^b - V(q) - \Delta V(q) \right] \mathrm{d}t \right\} \\ &\equiv \lim_{N\to\infty} \left(\frac{m}{2\pi \mathrm{i} \epsilon \hbar} \right)^{ND/2} \prod_{j=1}^{N-1} \int \mathrm{d}q^{(j)} \sqrt{g(q^{(j)})} \\ &\quad \times \exp\left\{ \frac{\mathrm{i}}{\hbar} \sum_{j=1}^{N} \left[\frac{m}{2\epsilon} h_{bc}(q^{(j)}) h_{ac}(q^{(j-1)}) \Delta q^{(j),a} \Delta q^{(j),b} \right.\right. \\ &\quad \left.\left. - \epsilon V(q^{(j)}) - \epsilon \Delta V(q^{(j)}) \right] \right\}. \end{aligned} \tag{2.6}$$

Here $\Delta q^{(j)} = q^{(j)} - q^{(j-1)}$ for $q^{(j)} = q(t' + j\epsilon)$ ($\epsilon = (t'' - t')/N = T/N$, $j = 1, \ldots, N$). ΔV denotes the well-defined quantum potential

$$\begin{aligned} \Delta V = &\frac{\hbar^2}{8m} \left[g^{ab} \Gamma_a \Gamma_b + 2(g^{ab}\Gamma_b)_{,b} + g^{ab}{}_{,ab} \right] \\ &+ \frac{\hbar^2}{8m} \left(2h^{ac} h^{bc}{}_{,ab} - h^{ac}{}_{,a} h^{bc}{}_{,b} - h^{ac}{}_{,b} h^{bc}{}_{,a} \right) \end{aligned} \tag{2.7}$$

arising from the specific lattice formulation for the path integral, respectively, the ordering prescription for position and momentum operators in equation (2.5). In this paper we always use the lattice prescription from equation (2.6).

In order to discuss time transformations in path integrals we consider the D-dimensional path integral

$$K(x'',x';T)=\lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon\hbar}\right)^{ND/2}\prod_{j=1}^{N-1}\int f^D(x^{(j)})\mathrm{d}x^{(j)}$$
$$\times\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}f(x^{(j-1)})f(x^{(j)})(x^{(j)}-x^{(j-1)})^2-\epsilon\frac{V(x^{(j)})}{f^2(x^{(j)})}\right]\right\}. \tag{2.8}$$

We perform the time transformation

$$s=\int_{t'}^{t}\frac{\mathrm{d}\sigma}{f^2[x(\sigma)]}\qquad s''=s(t'')\qquad s(t')=0\qquad \int_0^{s''}f^2[x(s)]\mathrm{d}s=T \tag{2.9}$$

where the lattice interpretation reads $\epsilon/[f(x^{(j-1)})f(x^{(j)})]=\delta^{(j)}\equiv\delta$. We identify $x(t)\equiv x[s(t)]$. The transformation formulae for a pure time transformation are, according to Duru and Kleinert [30] and Kleinert [31], given by

$$\begin{aligned}K(x'',x';T)&=\frac{1}{2\pi i\hbar}\int_{-\infty}^{\infty}\mathrm{d}E\,\mathrm{e}^{-iET/\hbar}G(x'',x';E)\\ G(x'',x';E)&=i[f(x')f(x'')]^{1-D/2}\int_0^{\infty}\mathrm{d}s''\,\tilde{K}(x'',x';s'')\end{aligned} \tag{2.10}$$

where the transformed path integral $\tilde{K}(s'')$ has the form

$$\tilde{K}(x'',x';s'')=\int_{x(0)=x'}^{x(s'')=x''}\mathcal{D}x(s)\exp\left\{\frac{i}{\hbar}\int_0^{s''}\left[\frac{m}{2}\dot{x}^2-V(x)+f^2(x)E\right]\mathrm{d}s\right\}. \tag{2.11}$$

A product lattice formulation is assumed in the path integral (2.11). Note the difference in comparison with a combined spacetime transformation, where a factor $[f(z')f(z'')]^{1/4}$ would appear instead [28, 30].

Finally, I cite the separation technique in path integrals [32]. Let us assume that the potential problem $V(x)$ has an exact solution according to

$$\int_{x(t')=x'}^{x(t'')=x''}\mathcal{D}x(t)\exp\left\{\frac{i}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{x}^2-V(x)\right]\mathrm{d}t\right\}$$
$$=\int\mathrm{d}E_\lambda\,\mathrm{e}^{-iE_\lambda T/\hbar}\Psi_\lambda^*(x')\Psi_\lambda(x''). \tag{2.12}$$

Here $\int \mathrm{d}E_\lambda$ denotes a Lesbeque–Stieljes integral to include discrete as well as continuous states. Now we consider the path integral

$$
\begin{aligned}
&K(z'',z',x'',x';T)\\
&\quad=\lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\epsilon\hbar}\right)^{ND/2}\prod_{j=1}^{N-1}\int f^d(z^{(j)})\prod_{i=1}^{d'}g_i(z^{(j)})\mathrm{d}z_i^{(j)}\int\prod_{k=1}^{d}\mathrm{d}x_k^{(j)}\\
&\qquad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}\left(\sum_{i=1}^{d'}g_i(z^{(j-1)})g_i(z^{(j)})\Delta^2 z_i^{(j)}\right.\right.\right.\\
&\qquad\left.+f(z^{(j-1)})f(z^{(j)})\sum_{k=1}^{d}\Delta^2 x_k^{(j)}\right)\\
&\qquad\left.\left.-\epsilon\left(\frac{V(x^{(j)})}{f^2(z^{(j)})}+W(z^{(j)})+\Delta W(z^{(j)})\right)\right]\right\}.
\end{aligned}
\tag{2.13}
$$

Here $(z,x)\equiv(z_i,x_k)$ $(i=1,\dots,d';k=1,\dots,d,\ d'+d=D)$ denote a D-dimensional coordinate system, g_i and f the corresponding metric terms, and ΔW a quantum potential according to equation (2.7). For simplicity we assume that the metric tensor g_{ab} involved has only diagonal elements, that is, $g_{ab}=\mathrm{diag}[g_1^2(z),g_2^2(z),\dots,g_{d'}^2(z),f^2(z),\dots,f^2(z)]$. $\mathrm{Det}(g_{ab})=f^{2d}\prod_{i=1}^{d'}g_i^2\equiv f^{2d}\,\hat{g}(z)$. The indices i and k will be omitted in the following. As shown in [32] the x path integration can be separated by performing a time transformation according to (2.9) forth and back yielding

$$
\begin{aligned}
&K(z'',z',x'',x';T)\\
&\quad=[f(z')f(z'')]^{-d/2}\int\mathrm{d}E_\lambda\Psi_\lambda^*(x')\Psi_\lambda(x'')\int\limits_{z(t')=z'}^{z(t'')=z''}\sqrt{\hat{g}(z)}\mathcal{D}z(t)\\
&\qquad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}g^2(z)\dot{z}^2-W(z)-\Delta W(z)-\frac{E_\lambda}{f^2(z)}\right]\mathrm{d}t\right\}.
\end{aligned}
\tag{2.14}
$$

We will rely strongly on these formulae in this paper. The limits $x(t')=x'$ and $x(t'')=x''$ etc will be omitted in the following.

3. Quantum motion on the hyperbolic space $\mathcal{H}^n$

We start by considering the n-dimensional generalization $\mathcal{H}^n$ of the Poincaré upper half-plane $\mathcal{H}$ [13–20]

$$
\mathcal{H}^n=\{(x_1,\dots,x_{n-1},y)|x_1,\dots,x_{n-1}\in\mathbb{R},y>0\}
\tag{3.1}
$$

endowed with the hyperbolic geometry (1.2). We call this parametrization the rectangular coordinate system for $\mathcal{H}^n$. The metric tensor and its inverse are given by

$$(g_{ab}) = \frac{1}{y^2}\mathrm{diag}(\underbrace{1,\ldots,1}_{D-1\ \text{times}}) \qquad (g^{ab}) = y^2\mathrm{diag}(\underbrace{1,\ldots,1}_{D-1\ \text{times}}) \tag{3.2}$$

and its determinant is $g = \det(g_{ab}) = y^{-2n} = y^{-2(D-1)}$. The invariant volume element reads as $\mathrm{d}V = \mathrm{d}x_1 \ldots \mathrm{d}x_k \mathrm{d}y/y^{D-1}$. The classical Lagrangian and Hamiltonian have the form respectively

$$\mathcal{L}_{\mathrm{Cl}} = \frac{m}{2}\frac{\dot{x}_1^2 + \cdots + \dot{x}_{n-1}^2 + \dot{y}^2}{y^2} \qquad \mathcal{H}_{\mathrm{Cl}} = \frac{1}{2m}y^2(p_{x_1}^2 + \cdots + p_{x_{n-1}}^2 + p_y^2). \tag{3.3}$$

According to the prescription given in section 2, the construction of the momentum operators and the quantum Hamiltonian is straightforward yielding for the momentum operators

$$p_{x_k} = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x_k} \qquad (k = 1,\ldots,D-1) \qquad p_y = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial y} - \frac{D-1}{2y}\right) \tag{3.4}$$

and for the quantum Hamiltonian we obtain

$$\begin{aligned} H &= -\frac{\hbar^2}{2m}\frac{1}{\sqrt{g}}\frac{\partial}{\partial q^a}g^{ab}\sqrt{g}\frac{\partial}{\partial q^b} \\ &= -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_{D-2}^2} + \frac{\partial^2}{\partial y^2} - \frac{D-3}{y}\frac{\partial}{\partial y}\right) \\ &= \frac{1}{2m}y(p_{x_1}^2 + \cdots + p_{x_{D-2}}^2 + p_y^2)y + \Delta V(y) \end{aligned} \tag{3.5}$$

with the quantum potential $\Delta V(y)$ given by

$$\Delta V(y) = \frac{\hbar^2}{8m}(D-1)(D-3). \tag{3.6}$$

Consequently the path integral for the free quantum motion on $\mathcal{H}^n$ has the form

$$\begin{aligned} &K^{\mathcal{H}^n}(\{x'',x'\},y'',y';T) \\ &= \int \mathcal{D}\{x(t)\}\frac{\mathcal{D}y(t)}{y^{D-1}}\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\frac{\dot{x}_1^2 + \cdots + \dot{x}_{D-2}^2 + \dot{y}^2}{y^2}\right.\right. \\ &\qquad \left.\left. - \frac{\hbar^2}{8m}(D-1)(D-3)\right]\mathrm{d}t\right\} \\ &= \lim_{N\to\infty}\left(\frac{m}{2\pi\mathrm{i}\epsilon\hbar}\right)^{N(D-1)/2}\prod_{j=1}^{N-1}\left(\int_0^\infty \frac{\mathrm{d}y_{(j)}}{y_{(j)}^{D-1}}\prod_{k=1}^{D-2}\int_{-\infty}^{\infty}\mathrm{d}x_{(j),k}\right) \\ &\qquad \times \exp\left\{\frac{\mathrm{i}}{\hbar}\sum_{j=1}^{N}\left[\frac{m}{2\epsilon}\frac{\Delta^2 x_{(j),1} + \cdots + \Delta^2 x_{(j),D-2} + \Delta^2 y_{(j)}}{y_{(j-1)}y_{(j)}}\right.\right. \\ &\qquad \left.\left. -\epsilon\frac{\hbar^2}{8m}(D-1)(D-3)\right]\right\} \end{aligned} \tag{3.7}$$

and we use once and for all the product lattice formulation. $\{x\}$ denotes the collection of the variables x_k $(k = 1,\ldots,D-2)$. We evaluate this path integral in three alternative ways.

3.1. Calculation of the Green function

We perform a time transformation in the path integral (3.7)

$$s(t) = \int_{t'}^{t} y^2(s)\mathrm{d}s \qquad s'' = s(t'') \qquad s(t') = 0 \qquad \int_0^{s''} \frac{\mathrm{d}s}{y^2(s)} = T. \tag{3.8}$$

This gives the transformation formulae

$$\begin{aligned} K^{\mathcal{H}^n}(\{x'',x'\},y'',y';T) &= \frac{1}{2\pi\mathrm{i}\hbar}\int_{-\infty}^{\infty} G^{\mathcal{H}^n}(\{x'',x'\},y'',y';E)\mathrm{e}^{-\mathrm{i}ET/\hbar}\,\mathrm{d}E \\ G^{\mathcal{H}^n}(\{x'',x'\},y'',y';\tilde{E}) &= \mathrm{i}(y'y'')^{(D-3)/2}\int_0^{\infty} \tilde{K}^{\mathcal{H}^n}(\{x'',x'\},y'',y';s'')\,\mathrm{d}s'' \end{aligned} \tag{3.9}$$

where we have abbreviated $\tilde{E} = E - (\hbar^2/8m)(D-1)(D-3)$ and the transformed kernel $\tilde{K}(s'')$ is given by

$$\begin{aligned} &\tilde{K}^{\mathcal{H}^n}(\{x'',x'\},y'',y';s'') \\ &\quad= \int \mathcal{D}\{x(s)\}\mathcal{D}y(s)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}(\dot{x}_1^2+\cdots+\dot{x}_{D-2}^2+\dot{y}^2)+\frac{\tilde{E}}{y^2}\right]\mathrm{d}t\right\} \\ &\quad\equiv \int \mathcal{D}\{x(s)\}\mu_\lambda[y^2]\mathcal{D}y(s)\exp\left[\frac{\mathrm{i}m}{2\hbar}\int_{t'}^{t''}(\dot{x}_1^2+\cdots+\dot{x}_{D-2}^2+\dot{y}^2)\mathrm{d}t\right] \\ &\quad= 2\pi\sqrt{y'y''}\left(\frac{m}{2\pi\mathrm{i}\hbar s''}\right)^{D/2} \\ &\qquad\times\exp\left[-\frac{m}{2\mathrm{i}\hbar s''}\left(\sum_{k=1}^{D-2}(x_k''-x_k')^2+y'^2+y''^2\right)\right] I_\lambda\left(\frac{m}{\mathrm{i}\hbar s''}y'y''\right). \end{aligned} \tag{3.10}$$

Here $\lambda = \pm\mathrm{i}\sqrt{2mE/\hbar^2-(D-2)^2/4}$, and we have applied the well-known path integral identity [33–35]

$$\begin{aligned} &\int\limits_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t)\mu_\nu[r^2]\exp\left[\frac{\mathrm{i}m}{2\hbar}\int_{t'}^{t''}(\dot{r}^2-\omega^2r^2)\,\mathrm{d}t\right] \\ &\qquad= \frac{m\omega\sqrt{r'r''}}{\mathrm{i}\hbar\sin\omega T}\exp\left[-\frac{m\omega}{2i\hbar}(r'^2+r''^2)\cot\omega T\right] I_\nu\left(\frac{m\omega r'r''}{\mathrm{i}\hbar\sin\omega T}\right) \end{aligned} \tag{3.11}$$

for the radial harmonic oscillator with the functional weight $\mu_\nu[r^2]$

$$\begin{aligned} \mu_\nu[r^2] &= \lim_{N\to\infty}\prod_{j=1}^{N}\mu_\nu[r^{(j-1)}r^{(j)}] \\ &= \lim_{N\to\infty}\prod_{j=1}^{N}\left(\frac{2\pi m r^{(j-1)}r^{(j)}}{\mathrm{i}\epsilon\hbar}\right)^{1/2}\exp\left(-\frac{m r^{(j-1)}r^{(j)}}{\mathrm{i}\epsilon\hbar}\right) I_\nu\left(\frac{m r^{(j-1)}r^{(j)}}{\mathrm{i}\epsilon\hbar}\right) \end{aligned} \tag{3.12}$$

in order to guarantee a well-defined short-time kernel [28, 35, 36]. Introducing the hyperbolic distance [16]

$$\cosh d(q'',q') = \frac{\sum_{k=1}^{D-2}(x_k''-x_k')^2 + y'^2 + y''^2}{y'y''} \tag{3.13}$$

we obtain for the Green function

$$\begin{aligned} G^{\mathcal{H}^n}(\{x'',x'\},y'',y';E) &= \frac{m}{\hbar}(2\pi)^{1-D/2} \\ &\quad\times \int_0^\infty z^{D/2-2}\exp\{-z\cosh d(q'',q')\} I_\lambda(z)\mathrm{d}z \\ &= \frac{m}{\hbar}(2\pi)^{1-D/2}\frac{\Gamma(\frac{1}{2}D-1+\lambda)}{(\sinh d)^{D/2-1}}\mathcal{P}^{-\lambda}_{D/2-2}(\coth d) \\ &= \frac{m}{\pi\hbar}\left(\frac{\mathrm{e}^{-\mathrm{i}\pi}}{2\pi\sinh d}\right)^{(D-3)/2} \mathcal{Q}^{(D-3)/2}_{-\frac{1}{2}-\mathrm{i}\sqrt{2m(E-E_0)}/\hbar}(\cosh d). \end{aligned} \tag{3.14}$$

Here $E_0 = \hbar^2(D-2)^2/8m$, and $\mathcal{P}^\mu_\nu$, $\mathcal{Q}^\mu_\nu$ denote Legendre functions of the first and second kind, respectively. A Wick transformation has been performed and use has been made of the integral [37, p 713]

$$\begin{aligned} &\int_0^\infty \mathrm{e}^{-tz/\sqrt{z^2-1}} I_\mu(t)t^\nu\,\mathrm{d}t = \Gamma(\nu+\mu+1)(z^2-1)^{(\nu+1)/2}\mathcal{P}^{-\mu}_\nu(z) \\ &\mathrm{Re}(\tfrac{1}{2}D-2+\lambda) > -1 \end{aligned} \tag{3.15}$$

and the relations [37, p 1006]

$$\mathrm{e}^{-\mathrm{i}\pi\mu}\mathcal{Q}^\mu_\nu(\cosh z) = \sqrt{\frac{\pi}{2\sinh z}}\Gamma(\nu+\mu+1)\mathcal{P}^{-\nu-1/2}_{-\mu-1/2}(\coth z) \tag{3.16a}$$

$$\mathcal{Q}^{-\mu}_\nu(z) = \mathrm{e}^{-2\mathrm{i}\pi\mu}\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\mathcal{Q}^\mu_\nu(z). \tag{3.16b}$$

Note that we have chosen the negative sign in the square-root expression in order to get the correct retarded Green function. In particular, in order to work with well-defined mathematical formulae we assume that E has a small positive imaginary part $\mathrm{i}\epsilon$ and write $E+\mathrm{i}\epsilon$ (with real E and $\epsilon > 0$) instead of E whenever necessary. The result (3.14) is in accordance with [16, 38]. Furthermore, this result makes it possible to come into contact with the path integral and Green function for the D-dimensional pseudosphere Λ^{D-1} [38]. As has been shown in [38] the Green function for the D-dimensional pseudosphere parametrized by D-dimensional pseudopolar coordinates [39]

$$\begin{aligned} x_n &= \cosh\tau \\ x_{n-1} &= \sinh\tau\cos\theta_{n-2} \\ x_{n-2} &= \sinh\tau\sin\theta_{n-2}\cos\theta_{n-3} \\ &\;\vdots \\ x_2 &= \sinh\tau\ldots\sin\theta_2\cos\theta_1 \\ x_1 &= \sinh\tau\ldots\sin\theta_2\sin\theta_1 \end{aligned} \tag{3.17}$$

$(0 \leqslant \theta_j \leqslant \frac{1}{2}\pi,\ j = 1, \dots, n-2,\ \tau > 0)$ is given by equation (3.14) and the corresponding Feynman kernel has the form $[K^{\mathcal{H}^n}(T) \equiv K^{\Lambda^{D-1}}(T)]$:

$$K^{\mathcal{H}^n}(\{x'', x'\}, y'', y'; T) = \frac{1}{2\pi}\left(\frac{1}{2\pi \sinh d}\right)^{(D-3)/2} \int_0^\infty \mathrm{d}p \left|\frac{\Gamma(\mathrm{i}p + (D-2)/2)}{\Gamma(\mathrm{i}p)}\right|^2 \times \mathcal{P}^{(3-D)/2}_{\mathrm{i}p-1/2}(\cosh d) \exp\left[-\frac{\mathrm{i}\hbar T}{2m}\left(p^2 + \frac{(D-2)^2}{4}\right)\right]. \tag{3.18}$$

In particular for D odd one uses the general property

$$\mathcal{Q}^m_\nu(z) = (z^2-1)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}z^m} \mathcal{Q}_\nu(z)$$

together with the integral representation [37, p 819]

$$\mathcal{P}_{\nu-\frac{1}{2}}(z) = \int_0^\infty \frac{p \tanh \pi p \mathrm{d}p}{\nu^2 + p^2} \mathcal{P}_{\mathrm{i}p-1/2}(z). \tag{3.19}$$

For D even one has to use the relations (compare [38] for more details, there is a sign missing in some formulae,

$$\left(\frac{1}{\sinh z}\right)^{(D-3)/2} \left|\frac{\Gamma(\mathrm{i}p + (D-2)/2)}{\Gamma(\mathrm{i}p)}\right|^2 \mathcal{P}^{(3-D)/2}_{\mathrm{i}p-1/2}(\cosh z) = \left(-\frac{\mathrm{d}}{\mathrm{d}\cosh z}\right)^{(D-2)/2} \mathrm{e}^{\mathrm{i}pz} \tag{3.20a}$$

$$\left(\frac{1}{2\pi \sinh z}\right)^{n-1/2} \mathcal{Q}^{n-1/2}_{-\mathrm{i}p-1/2}(\cosh z) = \frac{\pi}{p}\left(\frac{\mathrm{d}}{2\pi \mathrm{d}\cosh z}\right)^n \mathrm{e}^{\mathrm{i}pz}. \tag{3.20b}$$

From equation (3.18) we deduce that the spectrum for the free motion on $\mathcal{H}^n$ is given by

$$E_p^{\mathcal{H}^n} = \frac{\hbar^2}{2m}\left[p^2 + \frac{(D-2)^2}{4}\right] \tag{3.21}$$

and the largest lower bound (zero-point energy) of the spectrum is

$$E_0^{\mathcal{H}^n} = \frac{\hbar^2 (D-2)^2}{8m} \tag{3.22}$$

which is in accordance with [16, 38].

3.2. Calculation of the wavefunctions in the rectangular coordinate system

We investigate the path integral on $\mathcal{H}^n$ by the separation technique. The x_k dependences ($k = 1, \ldots, D-2$) are that of free particles in $\mathbb{R}^{D-2}$ and we obtain by separating them ($k^2 = \sum_{l=1}^{D-2} k_l^2$)

$$\begin{aligned} K^{\mathcal{H}^n}(\{x'', x'\}, y'', y'; T) &= \exp\left[-\frac{\mathrm{i}\hbar T}{8m}(D-1)(D-3)\right](y'y'')^{(D-2)/2}\prod_{l=1}^{D-2}\int\frac{\mathrm{d}k_l}{2\pi}\mathrm{e}^{\mathrm{i}k_l(x_l''-x_l')} \\ &\quad\times\int\frac{\mathcal{D}y(t)}{y}\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\frac{\dot{y}^2}{y^2}-\frac{y^2}{2m}\hbar^2k^2\right]\mathrm{d}t\right\} \\ &= (y'y'')^{(D-2)/2}\prod_{l=1}^{D-2}\int\frac{\mathrm{d}k_l}{2\pi}\mathrm{e}^{\mathrm{i}k_l(x_l''-x_l')}\frac{2}{\pi^2}\int_0^\infty\mathrm{d}p \\ &\quad\times\exp\left[-\frac{\mathrm{i}\hbar T}{2m}\left(p^2+\frac{(D-2)^2}{4}\right)\right]p\sinh\pi p K_{\mathrm{i}p}(ky')K_{\mathrm{i}p}(ky'') \end{aligned} \tag{3.23}$$

where use has been made of the path integral solution for Liouville quantum mechanics [40], i.e. we have performed the coordinate transformation $y = r^q$ yielding [40, 41]

$$\begin{aligned} \int\frac{\mathcal{D}y(t)}{y}&\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\frac{\dot{y}^2}{y^2}-\frac{y^2}{2m}\hbar^2k^2\right]\mathrm{d}t\right\} \\ &= \mathrm{e}^{-\mathrm{i}\hbar T/8m}\int\mathcal{D}q(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{q}^2-\frac{\hbar^2k^2}{2m}e^{2q}\right]\mathrm{d}t\right\} \\ &= \frac{2}{\pi^2}\int_0^\infty\mathrm{d}p\exp\left[-\frac{\mathrm{i}\hbar T}{2m}\left(p^2+\frac{1}{4}\right)\right]p\sinh\pi pK_{\mathrm{i}p}(k\mathrm{e}^{q'})K_{\mathrm{i}p}(k\mathrm{e}^{q''}) \end{aligned} \tag{3.24}$$

accompanied by a careful Taylor expansion in the kinetic energy term of the path integral, i.e.

$$\frac{\mathrm{i}m}{2\epsilon\hbar}\frac{(y^{(j)}-y^{(j-1)})^2}{y^{(j-1)}y^{(j)}} \doteq \frac{\mathrm{i}m}{2\epsilon\hbar}(q^{(j)}-q^{(j-1)})^2-\frac{\mathrm{i}\epsilon\hbar}{8m} \tag{3.25}$$

where the symbol '$\doteq$' has been used—following DeWitt [26]—to 'denote equivalence as far as use in the path integral is concerned'. K_μ denotes a modified Bessel function. This gives the wavefunctions on $\mathcal{H}^n$

$$\Psi^{\mathcal{H}^n}_{\{k\},p}(\{x\}, y) = \left(\prod_{l=1}^{D-2}\frac{dk_l}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}k_lx_l}\right)\frac{1}{\pi}\sqrt{2p\sinh\pi p}\,y^{(D-2)/2}K_{\mathrm{i}p}(ky) \tag{3.26}$$

and the energy spectrum has already been given in equation (3.21). The orthonormality and completeness of these wavefunctions have been shown in [40].

The representations (3.14) and (3.23) can be transformed into each other. We consider the Green function from the kernel representation (3.23) and introduce $(D-2)$-dimensional polar coordinates for the scalar product

$$\boldsymbol{k}\cdot(\boldsymbol{x}''-\boldsymbol{x}')=|\boldsymbol{k}|\cdot|\boldsymbol{x}''-\boldsymbol{x}'|\cos\psi$$

with $\psi=\angle(\boldsymbol{k}\cdot(\boldsymbol{x}''-\boldsymbol{x}'))$. This gives $(k^2=|\boldsymbol{k}|^2)$

$$\begin{aligned}
G^{\mathcal{H}^n}(\{x'',x'\},y'',y';E)
&=\frac{2\hbar}{\pi^2}\int_0^\infty\frac{p\sinh\pi p\mathrm{d}p}{\hbar^2p^2/2m+E_0-E}(y'y'')^{(D-2)/2}\\
&\quad\times\left(\prod_{l=1}^{D-2}\int\frac{\mathrm{d}k_l}{2\pi}\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}''-\boldsymbol{x}')}\right)K_{\mathrm{i}p}(ky')K_{\mathrm{i}p}(ky'')\\
&=\frac{2\hbar}{\pi^2}\int_0^\infty\frac{p\sinh\pi p\mathrm{d}p}{\hbar^2p^2/2m+E_0-E}(y'y'')^{(D-2)/2}\\
&\quad\times(2\pi)^{2-D}\int_0^\infty k^{D-3}\mathrm{d}k\int\mathrm{d}\Omega_k\,\mathrm{e}^{\mathrm{i}k|\boldsymbol{x}''-\boldsymbol{x}'|\cos\psi}K_{\mathrm{i}p}(ky')K_{\mathrm{i}p}(ky'')\\
&=\frac{\hbar}{\pi^3}\left(\frac{1}{2\pi|x''-x'|}\right)^{(D-4)/2}(y'y'')^{(D-2)/2}\int_0^\infty\frac{p\sinh\pi p\,\mathrm{d}p}{\hbar^2p^2/2m+E_0-E}\\
&\quad\times\int_0^\infty\mathrm{d}k k^{(D-2)/2}J_{(D-4)/2}(k|\boldsymbol{x}''-\boldsymbol{x}'|)K_{\mathrm{i}p}(ky')K_{\mathrm{i}p}(ky'')
\end{aligned}\tag{3.27}$$

where the integral over $\mathrm{d}\Omega$ was, for example, calculated in [28]. Using now [37, p 696]

$$\begin{aligned}
&\int_0^\infty x^{\nu+1}J_\nu(cx)K_\mu(ax)K_\mu(bx)\mathrm{d}x\\
&\qquad=\frac{\sqrt{\pi}\,c^\nu\Gamma(\nu+\mu+1)\Gamma(\nu-\mu+1)}{2^{3/2}(ab)^{\nu+1}(u^2-1)^{(\nu+1/2)/2}}\mathcal{P}_{\mu-1/2}^{-\nu-1/2}(u)
\end{aligned}\tag{3.28}$$

where $u=(a^2+b^2+c^2)/2ab$, we obtain

$$\begin{aligned}
G^{\mathcal{H}^n}(\{x'',x'\},y'',y';E)&=\frac{\hbar}{2\pi}\left(\frac{1}{2\pi\sinh d}\right)^{(D-3)/2}\\
&\quad\times\int_0^\infty\frac{\mathrm{d}p}{\hbar^2p^2/2m+E_0-E}\left|\frac{\Gamma(\mathrm{i}p+(D-2)/2)}{\Gamma(\mathrm{i}p)}\right|^2\mathcal{P}_{\mathrm{i}p-1/2}^{(3-D)/2}(\cosh d)
\end{aligned}\tag{3.29}$$

and $\cosh d$ denotes the hyperbolic distance in $\mathcal{H}^n$. The last equation is the energy Fourier transformed from equation (3.18) and therefore it (compare [38]) equals (3.14).

3.3. Calculation of the wavefunctions in polar coordinates

We can also exploit the natural symmetry of the system. Introducing $(D-2)$-dimensional polar coordinates [42]:

$$\begin{aligned} x_1 &= r\cos\theta_1 \\ x_2 &= r\sin\theta_1\cos\theta_2 \\ x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3 \\ &\vdots \\ x_{D-3} &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{D-4}\cos\phi \\ x_{D-2} &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{D-4}\sin\phi \end{aligned} \tag{3.30}$$

where $0\leqslant\theta_\nu\leqslant\pi$ $(\nu=1,\ldots,D-4)$, $0\leqslant\phi\leqslant 2\pi$ and $r\geqslant 0$, we get

$$\begin{aligned} &K^{\mathcal{H}^n}(\{x'',x'\},y'',y';T) \\ &\quad= \exp\left[-\frac{\mathrm{i}\hbar T}{8m}(D-1)(D-3)\right] \\ &\qquad\times (y'y'')^{(D-3)/2}(r'r'')^{(3-D)/2}\sum_{l=0}^{\infty}\sum_{\mu} S_l^\mu(\Omega')S_l^\mu(\Omega'') \\ &\qquad\times\int\frac{\mathcal{D}y(t)}{y^2}\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\frac{\dot y^2+\dot r^2}{y^2}-\hbar^2y^2\frac{(l+(D-4)/2)^2-\frac{1}{4}}{2mr^2}\right]\mathrm{d}t\right\} \\ &\quad= \exp\left[-\frac{\mathrm{i}\hbar T}{8m}(D-1)(D-3)\right](y'y'')^{(D-2)/2}(r'r'')^{(4-D)/2} \\ &\qquad\times\sum_{l=0}^{\infty}\int_0^\infty \mathrm{d}k\, J_{l+(D-4)/2}(kr')J_{l+(D-4)/2}(kr'') \\ &\qquad\times\sum_{\mu}S_l^\mu(\Omega')S_l^\mu(\Omega'')\int\frac{\mathcal{D}y(t)}{y}\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\frac{\dot y^2}{y^2}-\frac{\hbar^2k^2}{2m}y^2\right]\mathrm{d}t\right\} \\ &\quad= (r'r'')^{(4-D)/2}\sum_{l=0}^{\infty}\int_0^\infty\mathrm{d}k\, J_{l+(D-4)/2}(kr')J_{l+(D-4)/2}(kr'') \\ &\qquad\times\sum_\mu S_l^\mu(\Omega')S_l^\mu(\Omega'')(y'y'')^{(D-2)/2} \\ &\qquad\times\frac{2}{\pi^2}\int_0^\infty\mathrm{d}p\exp\left[-\frac{\mathrm{i}\hbar T}{2m}\left(p^2+\frac{(D-2)^2}{4}\right)\right] \\ &\qquad\times p\sinh\pi p K_{\mathrm{i}p}(ky')K_{\mathrm{i}p}(ky'') \end{aligned} \tag{3.31}$$

where use has been made of equation (3.11) for $\omega\to 0$, [37, p 718]

$$\int_0^\infty \mathrm{e}^{-\rho^2x^2}J_p(ax)J_p(bx)x\,\mathrm{d}x = \frac{1}{2\rho^2}\exp\left(-\frac{a^2+b^2}{4\rho^2}\right)I_p\left(\frac{ab}{2\rho^2}\right) \tag{3.32}$$

and we have once again used the path integral solution for Liouville quantum mechanics. The S_l^μ are the real hyperspherical harmonics on the $(D-3)$-dimensional sphere $S^{(D-4)}$ [42] and Ω denotes a unit vector on $S^{(D-4)}$. The wavefunctions consequently are given by

$$\Psi_{k,l,p}^{\mathcal{H}^n}(r,\Omega,y) = r^{(4-D)/2}\sqrt{k}\,J_{l+(D-4)/2}(kr)S_l^\mu(\Omega)\frac{1}{\pi}\sqrt{2p\sinh\pi p}\,y^{(D-2)/2}K_{\mathrm{i}p}(ky). \tag{3.33}$$

3.4. *Incorporation of constant magnetic fields*

The incorporation of constant magnetic fields is quite easy and will be presented here for completeness. Generalizing properly the case for the Poincaré upper half-plane [43] we can formulate the path integral on $\mathcal{H}^n$ with the $(D-1)$-dimensional magnetic field vector $\boldsymbol{B}$ and vector potential $\boldsymbol{A}$, respectively,

$$\boldsymbol{A} = \frac{em}{2cy}\boldsymbol{B} \qquad \boldsymbol{b} = -\frac{em}{2c}\boldsymbol{B} \tag{3.34}$$

with e the electric charge and c the velocity of light. Similarly as in [44] we obtain for the appropriate path integral formulation ($A_g = B_g \equiv 0$ gauge)

$$\begin{aligned} K^{\mathcal{H}^n,b}(\{x'',x'\},y'',y';T) &= \exp\left[-\frac{\mathrm{i}\hbar T}{8m}(D-1)(D-3)\right]\int\frac{\mathcal{D}y(t)}{y^{D-1}}\int\mathcal{D}\{x(t)\} \\ &\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\frac{\dot{x}_1^2+\ldots+\dot{x}_{n-1}^2+\dot{y}^2}{y^2}\right.\right. \\ &\quad\left.\left.-\frac{b_1\dot{x}_1+\ldots+b_{D-2}\dot{x}_{D-2}}{y}\right]\mathrm{d}t\right\} \end{aligned} \tag{3.35}$$

and again the product lattice formulation is assumed. Performing a Fourier expansion in the x_l $(l = 1,\ldots,D-2)$ gives

$$K^{\mathcal{H}^n,b}(\{x'',x'\},y'',y';T) = \frac{1}{(2\pi)^{D-2}}\int_{-\infty}^{\infty}K_{\{k\}}^{\mathcal{H}^n,b}(y'',y';T)\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}''-\boldsymbol{x}')}d^{D-2}k \tag{3.36}$$

with

$$\begin{aligned} K_{\{k\}}^{\mathcal{H}^n,b}(y'',y';T) &= (y'y'')^{(D-2)/2}\exp\left[-\frac{\mathrm{i}\hbar T}{2m}\left(\frac{(D-2)^2}{4}+b^2\right)\right] \\ &\quad\times\int\mathcal{D}q(t)\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{q}^2-\frac{\hbar^2}{2m}(\boldsymbol{k}\cdot\boldsymbol{k}\,\mathrm{e}^{2q}-2\boldsymbol{k}\cdot\boldsymbol{b}\,\mathrm{e}^{q})\right]\mathrm{d}t\right\} \end{aligned} \tag{3.37}$$

where the transformation $y = \mathrm{e}^q$ has been made. The path integral in the coordinate q is now the path integral for the Morse potential

$$V^{\mathrm{M}} = \frac{\hbar^2 V_0^2}{2m}(\mathrm{e}^{2q} - 2\alpha \mathrm{e}^q) \tag{3.38}$$

which has been treated in [44–46]. Setting $\alpha = k \cdot b/|k|^2$ we get, therefore, the solution of the path integral (3.35) reading

$$\begin{aligned} K^{\mathcal{H}^n,b}(\{x'',x'\},y'',y';T) &= \sum_{n=0}^{N_M} \mathrm{e}^{-\mathrm{i}E_nT/\hbar}\Psi^*_{\{k\},n}(\{x'\},y')\Psi_{\{k\},n}(\{x''\},y'') \\ &\quad + \int_0^\infty \mathrm{d}p\,\mathrm{e}^{-\mathrm{i}E_pT/\hbar}\Psi^*_{\{k\},p}(\{x'\},y')\Psi_{\{k\},p}(\{x''\},y''). \end{aligned} \tag{3.39}$$

The wavefunctions and the energy spectrum for the discrete spectrum are

$$E_n = -\frac{\hbar^2}{8m}(2\alpha|k| - 2n - 1)^2 + \frac{\hbar^2}{2m}\left[b^2 + \frac{(D-2)^2}{4}\right]$$

$$n = 0, \ldots, N_M < \frac{k \cdot b}{|k|^2} - \frac{1}{2} \tag{3.40}$$

$$\Psi_{\{k\},n}(\{x\},y) = \frac{\mathrm{e}^{\mathrm{i}k\cdot x}}{(2\pi)^{(D-2)/2}}\sqrt{\frac{n!(2\alpha|k| - 2n - 1)}{2|k|\Gamma(\alpha|k| - n)}} \times (2|k|y)^{\alpha|k|-n}\mathrm{e}^{-|k|y}L_n^{(2\alpha|k|-2n-1)}(2|k|y) \tag{3.41a}$$

$$= \frac{\mathrm{e}^{\mathrm{i}k\cdot x}}{(2\pi)^{(D-2)/2}}\sqrt{\frac{n!(2\alpha|k| - 2n - 1)}{2|k|\Gamma(\alpha|k| - n)}} \times (-1)^n W_{\alpha|k|,\alpha|k|-n-1/2}(2|k|y). \tag{3.41b}$$

For the continuous spectrum we obtain

$$E_p = \frac{\hbar^2}{2m}\left[p^2 + b^2 + \frac{(D-2)^2}{4}\right] \tag{3.42}$$

$$\Psi_{\{k\},p}(\{x\},y) = \frac{\mathrm{e}^{\mathrm{i}k\cdot x}}{(2\pi)^{(D-2)/2}}\Gamma\left(\mathrm{i}p - \alpha|k| + \tfrac{1}{2}\right)\sqrt{\frac{p\sinh 2\pi p}{2\pi^2|k|}}\,W_{\alpha|k|,\mathrm{i}p}(2|k|y). \tag{3.43}$$

The $L_n^{(\lambda)}(z)$ and $W_{\mu,\nu}(z)$ denote Laguerre polynomials and Whittaker functions, respectively. The orthonormality and completeness properties of these wavefunctions have been, for example, discussed in [43, 44, 47, 48].

4. Quantum motion on the hyperbolic space SU$(n,1)$/S[U(1) × U(n)]

Next we consider the space S_2. It has a more complicated structure than S_1. Following Venkov [16] we take for the metric in the hyperbolic space $S_2 \cong \mathrm{SU}(n,1)/\mathrm{S}[\mathrm{U}(1)\times\mathrm{U}(n)]$

$$\begin{aligned} ds^2 &= \frac{dy^2}{y^2} + \frac{1}{y^2}\sum_{k=2}^{n} dz_k dz_k^* + \frac{1}{y^4}\left(dx_1 + \mathrm{Im}\sum_{k=2}^{n} z_k^* dz_k\right)^2 \\ &= \frac{dy^2}{y^2} + \frac{1}{y^2}\sum_{k=2}^{n}(dx_k^2 + dy_k^2) + \frac{1}{y^4}\left[dx_1^2 + 2dx_1\sum_{k=2}^{n}(x_k dy_k - y_k dx_k)\right. \\ &\qquad \left. + \left(\sum_{k=2}^{n}(x_k dy_k - y_k dx_k)\right)^2\right]. \end{aligned} \tag{4.1}$$

Here we have set $z_k = x_k + \mathrm{i}y_k \in \mathbb{C}$ for the $(n-1)$ complex variables. The metric tensor in the coordinates (4.1) is given by

$$(g_{ab}) = \frac{1}{y^4}\begin{pmatrix} 1 & -y_2 & x_2 & \dots & -y_n & x_n & 0 \\ -y_2 & y_2^2 + y^2 & -x_2 y_2 & \dots & y_2 y_2 & -y_2 x_n & 0 \\ x_2 & -x_2 y_2 & x_2^2 + y^2 & \dots & -x_2 y_n & x_2 x_n & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -y_n & y_2 y_n & -x_2 y_n & \dots & y_n^2 + y^2 & -x_n y_n & 0 \\ x_n & -x_n y_2 & x_2 x_n & \dots & -x_n y_n & x_n^2 + y^2 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & y^2 \end{pmatrix} \tag{4.2}$$

where we have ordered the matrix entries according to $(x_1, x_2, y_2, \dots, x_n, y_n, y) \times (x_1, x_2, y_2, \dots, x_n, y_n, y)$. Its inverse (g^{ab}) is calculated as

$$(g^{ab}) = y^2\begin{pmatrix} y^2 + \sum_{k=2}^{n}|z_k|^2 & y_2 & -x_2 & \dots & y_n & -x_n & 0 \\ y_2 & 1 & 0 & \dots & 0 & 0 & 0 \\ -x_2 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ y_n & 0 & 0 & \dots & 1 & 0 & 0 \\ -x_n & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}. \tag{4.3}$$

We call this parametrization the rectangular coordinate system for the space S_2. For the determinant we have $g = \det(g_{ab}) = 1/y^{4n+2}$. The determinant and (g^{ab}) can be obtained from (g_{ab}) by considering first the case $n = 2$ and then by induction. The hyperbolic distance in S_2 is measured by [16]

$$\begin{aligned} \cosh d(q', q'') = \frac{1}{4(y'y'')^2}&\left[\left(\sum_{k=1}^{D-2}(x_k'' - x_k')^2 + y'^2 + y''^2\right)^2\right. \\ &\left. + 4\left(x_1'' - x_1' + \sum_{k=2}^{n}(x_k'' y_k' - y_k'' x_k')\right)^2\right]. \end{aligned} \tag{4.4}$$

The classical Lagrangian and Hamiltonian, respectively, have the form

$$\mathcal{L}^{S_2} = \frac{m}{2}\left[\frac{\dot{y}^2}{y^2} + \frac{1}{y^2}\sum_{k=2}^{n}|\dot{z}_k|^2 + \frac{1}{y^4}\left(\dot{x}_1 + \Im\sum_{k=2}^{n} z_k^*\dot{z}_k\right)^2\right]$$

$$= \frac{m}{2}\left\{\frac{\dot{y}^2}{y^2} + \frac{1}{y^2}\sum_{k=2}^{n}(\dot{x}_k^2 + \dot{y}_k^2) + \frac{1}{y^4}\left[\dot{x}_1^2 + 2\dot{x}_1\sum_{k=2}^{n}(x_k\dot{y}_k - y_k\dot{x}_k)\right.\right.$$

$$\left.\left. + \left(\sum_{k=2}^{n}(x_k\dot{y}_k - y_k\dot{x}_k)\right)^2\right]\right\} \tag{4.5a}$$

$$\mathcal{H}^{S_2} = \frac{y^2}{2m}\left[p_y^2 + \sum_{k=2}^{n}(p_{x_k}^2 + p_{y_k}^2) + 2\sum_{k=2}^{n}(y_k p_{x_k} - x_k p_{y_k})p_{x_1}\right.$$

$$\left. + \left(y^2 + \sum_{k=2}^{n}(x_k^2 + y_k^2)\right)p_{x_1}^2\right]. \tag{4.5b}$$

According to the general theory we have

$$\begin{aligned} p_{x_k} &= \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x_k} \qquad k = 1,\dots,n \\ p_{y_k} &= \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial y_k} \qquad k = 2,\dots,n \\ p_y &= \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial y} - \frac{2n+1}{2y}\right). \end{aligned} \tag{4.6}$$

Therefore we obtain for the quantum Hamiltonian according to equation (2.5)

$$H^{S_2} = -\frac{\hbar^2}{2m}\Delta_{\mathrm{LB}}^{S_2} = -\frac{\hbar^2}{2m}y^2\left[\left(\frac{\partial^2}{\partial y^2} - \frac{2n-1}{y}\frac{\partial}{\partial y}\right) + \sum_{k=2}^{n}\left(\frac{\partial^2}{\partial {x_k}^2} + \frac{\partial^2}{\partial {x_k}^2}\right)\right.$$

$$\left. + 2\sum_{k=2}^{n}\left(y_k\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial y_k}\right)\frac{\partial}{\partial x_1} + \left(y^2 + \sum_{k=2}^{n}(x_k^2 + y_k^2)\right)\frac{\partial^2}{\partial x_1^2}\right]$$

$$= \frac{1}{2m}\left[y p_y^2 y + y^2\sum_{k=2}^{n}(p_{x_k}^2 + p_{y_k}^2) + 2y^2\sum_{k=2}^{n}(y_k p_{x_k} - x_k p_{y_k})p_{x_1}\right.$$

$$\left. + y^2\left(y^2 + \sum_{k=2}^{n}(x_k^2 + y_k^2)\right)p_{x_1}^2\right] + \Delta V(y) \tag{4.7}$$

with the quantum potential

$$\Delta V(y) = \frac{\hbar^2}{8m}(4n^2 - 1). \tag{4.8}$$

This gives for the path integral

$$
\begin{aligned}
&K^{S_2}(\{x'',y''\}_{k=2}^n,\{x',y'\}_{k=2}^n,x_1'',x_1',y'',y';T)\\
&\quad=\exp\left[-\frac{\mathrm{i}\hbar T}{8m}(4n^2-1)\right]\int\frac{\mathcal{D}y(t)}{y^{2n+1}}\int\mathcal{D}x_1(t)\prod_{k=2}^n\int\mathcal{D}x_k(t)\int\mathcal{D}y_k(t)\\
&\qquad\times\exp\left[\!\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\left\{\frac{1}{y^2}\left[\dot y^2+\sum_{k=2}^n(\dot x_k^2+\dot y_k^2)\right]\right.\right.\right.\right.\\
&\qquad+\frac{1}{y^4}\left[\dot x_1^2+2\dot x_1\sum_{k=2}^n(x_k\dot y_k-y_k\dot x_k)\right.\\
&\qquad\left.\left.\left.\left.+\left(\sum_{k=2}^n(x_k\dot y_k-y_k\dot x_k)\right)^2\right]\right\}\right)\mathrm{d}t\right]\!\right].
\end{aligned}
\tag{4.9}
$$

We evaluate this path integral in two alternative ways.

4.1. Calculation of the wavefunctions by $(n-1)$-fold two-dimensional polar coordinates

Introducing polar coordinates according to

$$
\begin{aligned}
x_k&=r_k\cos\phi_k\\
y_k&=r_k\sin\phi_k\qquad(r_k>0,0\leqslant\phi_k\leqslant 2\pi,k=2,\ldots,n)
\end{aligned}
\tag{4.10}
$$

yields for the path integral $K^{S_2}(T)$

$$
\begin{aligned}
&K^{S_2}(\{x'',y''\}_{k=2}^n,\{x',y'\}_{k=2}^n,x_1'',x_1',y'',y';T)\\
&\quad\equiv K^{S_2}(\{r_k'',r_k'\}_{k=2}^n,\{\phi_k'',\phi_k'\}_{k=2}^n,x_1'',x_1',y'',y';T)\\
&\quad=\exp\left[-\frac{\mathrm{i}\hbar T}{8m}(4n^2-1)\right]\int\frac{\mathcal{D}y(t)}{y^{2n+1}}\int\mathcal{D}x_1(t)\prod_{k=2}^n\int r\mathcal{D}r_k(t)\int\mathcal{D}\phi_k(t)\\
&\qquad\times\exp\left(\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left\{\frac{m}{2y^2}\left[\dot y^2+\sum_{k=2}^n(\dot r_k^2+r_k^2\dot\phi_k^2)\right]\right.\right.\\
&\qquad\left.\left.+\frac{m}{2y^4}\left[\dot x_1^2+2\dot x_1\sum_{k=2}^n r_k^2\dot\phi_k+\left(\sum_{k=2}^n r_k^2\dot\phi_k\right)^2\right]+\frac{\hbar^2y^2}{8m}\sum_{k=2}^n\frac{1}{r_k^2}\right\}\mathrm{d}t\right)
\end{aligned}
\tag{4.11}
$$

and an additional quantum potential

$$
\Delta V=-\sum_{k=2}^n\frac{\hbar^2y^2}{8mr_k^2}
\tag{4.12}
$$

appears due to these polar coordinates [28]. In order to evaluate the path integral (4.11) we start with a Fourier expansion in the coordinate x_1. This gives

$$
\begin{aligned}
&K^{S_2}(\{r_k'',r_k'\}_{k=2}^n,\{\phi_k'',\phi_k'\}_{k=2}^n,x_1'',x_1',y'',y';T)\\
&\quad=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k_1\,K_{k_1}^{S_2}(\{r_k'',r_k'\}_{k=2}^n,\{\phi_k'',\phi_k'\}_{k=2}^n,y'',y';T)\,\mathrm{e}^{\mathrm{i}k_1(x_1''-x_1')}
\end{aligned}
\tag{4.13}
$$

and we obtain for the kernel $K_{k_1}(T)$

$$K^{S_2}_{k_1}(\{r_k'', r_k'\}_{k=2}^n, \{\phi_k'', \phi_k'\}_{k=2}^n, y'', y'; T)$$
$$= \int_{-\infty}^{\infty} \mathrm{d}x_1'' \mathrm{e}^{-\mathrm{i}k_1(x_1''-x_1')} K(\{x'', y''\}_{k=2}^n, \{x', y'\}_{k=2}^n, x_1'', x_1', y'', y'; T)$$
$$= y'y'' \exp\left[-\frac{\mathrm{i}\hbar T}{8m}(4n^2-1)\right] \int \frac{\mathcal{D}y(t)}{y^{2n-1}} \prod_{k-2}^n \int r_k \mathcal{D}r_k(t) \mathcal{D}\phi_k(t)$$
$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2y^2}\left(\dot{y}^2 + \sum_{k=2}^n (\dot{r}_k^2 + r_k^2\dot{\phi}_k^2)\right)\right.\right.$$
$$\left.\left. -\frac{\hbar^2 y^4}{2m}k_1^2 + \hbar k_1 \sum_{k=2}^n r_k^2 \dot{\phi}_k + \sum_{k=2}^n \frac{\hbar^2 y^2}{8mr_k^2}\right]\mathrm{d}t\right\}. \tag{4.14}$$

Proceeding, we perform a Fourier expansion in the variables ϕ_k $(k = 2, \ldots, n)$, i.e.

$$K^{S_2}_{k_1}(\{r_k'', r_k'\}_{k=2}^n, \{\phi_k'', \phi_k'\}_{k=2}^n, y'', y'; T)$$
$$= \frac{1}{(2\pi)^{n-1}} \prod_{k=2}^n \sum_{l_k=-\infty}^{\infty} K_{k_1,\{l_k\}}(\{r_k'', r_k'\}_{k=2}^n, y'', y'; T)\, \mathrm{e}^{\mathrm{i}l_k(\phi_k''-\phi_k')} \tag{4.15}$$

with the kernel $K_{k_1,\{l_k\}}(T)$ given by

$$K^{S_2}_{k_1,\{l_k\}}(\{r_k'', r_k'\}_{k=2}^n, y'', y'; T) = \prod_{k=2}^n (r_k' r_k'')^{-1/2} (y'y'')^{(n+1)/2}$$
$$\times \exp\left[-\frac{i\hbar T}{8m}(4n^2-1)\right] \int \frac{\mathcal{D}y(t)}{y^n} \prod_{k-2}^n \int \mathcal{D}r_k(t)$$
$$\times \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2y^2}\left(\dot{y}^2 + \sum_{k=2}^n \dot{r}_k^2\right)\right.\right.$$
$$\left.\left. -\frac{\hbar^2 y^2}{2m}\left(k_1^2 \sum_{k=2}^n r_k^2 + \sum_{k=2}^n \frac{l_k^2 - \frac{1}{4}}{r_k^2} - 2\sum_{k=2}^n l_k k_1 + k_1^2 y^2\right)\right]\mathrm{d}t\right\}. \tag{4.16}$$

Here we have used that in the limit $\epsilon \to 0$ we can set $\int_0^{2\pi} \to \int_{-\infty}^{\infty}$ which is standard in path integration technique. We observe that the path integrations in the variables r_k $(k = 2, \ldots, n)$ are that of radial harmonic oscillators with angular number $\nu = l_k$ and frequency $\omega = \hbar|k_1|/m$ [cf equation (3.11)]. Therefore [33, 35]

$$K^{S_2}_{k_1,\{l_k\}}(\{r_k'', r_k'\}_{k=2}^n, y'', y'; T)$$
$$= (y'y'')^n \left(\prod_{k=2}^n (r_k' r_k'')^{-\frac{1}{2}} R_{n_k}^{l_k}(r_k') R_{n_k}^{l_k}(r_k'')\right) K^{S_2}_{k_1,\{l_k\, n_k\}}(y'', y'; T) \tag{4.17}$$

with the radial wavefunctions $R_n^l(r)$ given by

$$R_n^l(r) = \sqrt{\frac{2|k_1|n!}{\Gamma(n+|l|+1)}}\,(|k_1|r)^{|l|}\exp(-|k_1|r)L_n^{(|l|)}(|k_1|r^2) \quad (4.18)$$

and the remaining path integral

$$\begin{aligned} K^{S_2}_{k_1,\{l_k\,n_k\}}(y'',y';T) &= \exp\left[-\frac{\mathrm{i}\hbar T}{8m}(4n^2-1)\right]\int\frac{\mathcal{D}y(t)}{y} \\ &\quad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\frac{\dot y^2}{y^2}-\frac{\hbar^2k_1^2}{2m}y^4-E_\lambda y^2\right]\mathrm{d}t\right\}. \end{aligned} \quad (4.19)$$

The quantity E_λ is given by

$$E_\lambda = \frac{\hbar^2}{m}\sum_{k=2}^{n}[|k_1|(2n_k+1)+|k_1l_k|-k_1l_k] \quad (4.20)$$

and we have $E_\lambda > 0$. The path integral (4.19) is now of the type of oscillator-like potentials discussed in [32] which can be solved by means of the path integral for the Morse potential (cf section 3). Therefore we obtain ($\omega = \hbar|k_1|/m$)

$$\begin{aligned} K^{S_2}_{k_1,\{l_k,n_k\}}(y'',y';T) &= \frac{\hbar}{2\pi^2 m\omega y'y''}\int_0^\infty \mathrm{d}p\,p\sinh\pi p \\ &\quad\times\left|\Gamma\left[\frac{1}{2}\left(1+\mathrm{i}p+\frac{E_\lambda}{\hbar\omega}\right)\right]\right|^2\exp\left[-\frac{\mathrm{i}\hbar T}{2m}(p^2+n^2)\right] \\ &\quad\times W_{-E_\lambda/2\hbar\omega,\mathrm{i}p/2}\left(\frac{m\omega}{\hbar}y''^2\right)W_{-E_\lambda/2\hbar\omega,\mathrm{i}p/2}\left(\frac{m\omega}{\hbar}y'^2\right). \end{aligned} \quad (4.21)$$

Note that due to $E_\lambda > 0$ only a continuous spectrum is allowed [32]. Combining all relevant terms we obtain for the path integral solution on S_2

$$\begin{aligned} &K^{S_2}(\{x'',y''\}_{k=2}^n,\{x',y'\}_{k=2}^n,x_1'',x_1',y'',y';T) \\ &\quad\equiv K^{S_2}(\{r_k'',r_k'\}_{k=2}^n,\{\phi_k'',\phi_k'\}_{k=2}^n,x_1'',x_1',y'',y';T) \\ &\quad= \int_{-\infty}^{\infty}\mathrm{d}k_1\prod_{k=2}^{n}\sum_{l_k=-\infty}^{\infty}\sum_{n_k=0}^{\infty}\int_0^\infty \mathrm{d}p\,\Psi^{S_2\,*}_{k_1,\{l_k,n_k\},p}(\{r_k',\phi_k'\},x_1',y') \\ &\qquad\times\Psi^{S_2}_{k_1,\{l_k,n_k\},p}(\{r_k'',\phi_k''\},x_1'',y'')\,\mathrm{e}^{-\mathrm{i}TE_p/\hbar} \end{aligned} \quad (4.22)$$

with the energy spectrum

$$E_p = \frac{\hbar^2}{2m}(p^2+n^2) \quad (4.23)$$

and the wavefunctions

$$\Psi^{S_2}_{k_1,\{l_k,n_k\},p}(\{r_k,\phi_k\},x_1,y)=\frac{1}{(2\pi)^{n/2}}e^{ik_1x_1}e^{il_k\phi_k}\frac{1}{\sqrt{r_k}}R^{l_k}_{n_k}(r_k)\Phi_p(y) \tag{4.24}$$

where the $R^{l_k}_{n_k}(r_k)$ have been given in equation (4.18) and the $\Phi_p(y)$ have the form

$$\Phi_p(y)=\sqrt{\frac{p\sinh\pi p}{2\pi^2|k_1|}}\,\Gamma\left[\frac{1}{2}\left(1+ip+\frac{mE_\lambda}{|k_1|}\right)\right]y^{n-1}W_{-mE_\lambda/2|k_1|,ip/2}(|k_1|y^2). \tag{4.25}$$

In particular the energy spectrum with its largest lower bound is $E_0=\hbar^2n^2/2m$ which is in accordance with [16].

4.2. Calculation of the wavefunctions in $SU(n)$ polar coordinates

Due to the symmetry properties of the space S_2, we also can use $\mathrm{SU}(n)$ polar coordinates for the separation. We introduce [49, 50] $(2n-1)$-dimensional $\mathrm{SU}(n-1)$ polar coordinates according to

$$\begin{aligned}
z_n&=re^{i\phi_n}\cos\theta_{n-1}\\
z_{n-1}&=re^{i\phi_{n-1}}\sin\theta_{n-1}\cos\theta_{n-2}\\
z_{n-2}&=re^{i\phi_{n-2}}\sin\theta_{n-1}\sin\theta_{n-2}\cos\theta_{n-3}\\
&\vdots\\
z_3&=re^{i\phi_3}\sin\theta_{n-1}\ldots\sin\theta_3\cos\theta_2\\
z_2&=re^{i\phi_2}\sin\theta_{n-1}\ldots\sin\theta_3\sin\theta_2
\end{aligned} \tag{4.26}$$

with $0\leqslant\phi_i\leqslant2\pi, i=2,\ldots,n, 0\leqslant\theta_j\leqslant\pi/2, j=2,\ldots,n-1$ and $r\geqslant0$. In terms of these coordinates the free classical Lagrangian is given by

$$\begin{aligned}
\mathcal{L}_{\mathrm{Cl}}(z,\dot z)\equiv\mathcal{L}_{\mathrm{Cl}}(r,\dot r,\{\theta,\dot\theta\},\{\phi,\dot\phi\})&=\tfrac{1}{2}m\left(|\dot z_2|^2+|\dot z_3|^2+\ldots+|\dot z_n|^2\right)\\
&=\tfrac{1}{2}m\left(\dot r^2+r^2\left\{\dot\theta^2_{n-1}+\cos^2\theta_{n-1}\dot\phi^2_n+\sin^2\theta_{n-1}\right.\right.\\
&\quad\times\left[\dot\theta^2_{n-2}+\cos^2\theta_{n-2}\dot\phi^2_{n-1}+\ldots+\sin^2\theta_3\right.\\
&\quad\times\left.\left.\left.\left(\dot\theta^2_2+\cos^2\theta_2\dot\phi^2_3+\sin^2\theta_2\dot\phi^2_2\right)\ldots\right]\right\}\right).
\end{aligned} \tag{4.27}$$

The corresponding quantum potential has the form [50]

$$\begin{aligned}
\Delta V(r,\{\theta,\phi\})=-\frac{\hbar^2}{8m}&\left\{1+\frac{1}{\cos^2\theta_{n-1}}+\frac{1}{\sin^2\theta_{n-1}}\left[1+\frac{1}{\cos^2\theta_{n-2}}\right.\right.\\
&\left.\left.+\cdots+\frac{1}{\sin^2\theta_3}\left(1+\frac{1}{\cos^2\theta_2}+\frac{1}{\sin^2\theta_2}\right)\ldots\right]\right\}.
\end{aligned} \tag{4.28}$$

Transforming the path integral into these coordinates and performing the Fourier expansion with respect to x_1 thus yields

$$
\begin{aligned}
K_{k_1}^{S_2}(\{x'',y''\}_{k=2}^{n},\{x',y'\}_{k=2}^{n},y'',y';T) &\equiv K_{k_1}^{S_2}(r'',r',\{\theta'',\theta'\},\{\phi'',\phi'\},y'',y';T)\\
&= y'y''\exp\left[-\frac{\mathrm{i}\hbar T}{8m}(4n^2-1)\right]\int\frac{\mathcal{D}y(t)}{y^{2n-1}}\int r^{2n-3}\mathcal{D}r(t)\\
&\quad\times\int\prod_{k=2}^{n-1}\cos\theta_k(\sin\theta_k)^{2k-3}\mathcal{D}\theta_k(t)\int\prod_{j=2}^{n}\mathcal{D}\phi_j(t)\\
&\quad\times\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2y^2}\left\{\dot y^2+\dot r^2+r^2\left[\dot\theta_{n-1}^2+\cos^2\theta_{n-1}\dot\phi_n^2+\cdots\right.\right.\right.\right.\\
&\quad\left.\left.+\sin^2\theta_3\left(\dot\theta_3^2+\cos^2\theta_3\dot\phi_3^2+\sin^2\theta_2\dot\phi_2^2\right)\ldots\right]\right\}-\frac{\hbar^2k_1^2}{2m}y^4\\
&\quad+\hbar k_1r^2\left\{\dot\phi_n\cos^2\theta_{n-1}+\sin^2\theta_{n-1}\left[\dot\phi_{n-1}\sin^2\theta_{n-2}+\cdots\right.\right.\\
&\quad\left.\left.+\sin^2\theta_3(\dot\phi_3\cos^2\theta_2+\dot\phi_2\sin^2\theta_2)\ldots\right]\right\}+\frac{\hbar^2y^2}{8mr^2}\\
&\quad\left.\left.\times\left[1+\frac{1}{\cos^2\theta_{n-1}}+\cdots+\frac{1}{\sin^2\theta_3}\left(1+\frac{1}{\cos^2\theta_2}+\frac{1}{\sin^2\theta_2}\right)\ldots\right]\right)\mathrm{d}t\right].
\end{aligned}
\tag{4.29}
$$

In the appendix we evaluate the path integration over the $\mathrm{SU}(n-1)$ coordinates. Separating (4.29) with respect to these coordinates yields

$$
\begin{aligned}
&K_{k_1}^{S_2}(r'',r',\{\theta'',\theta'\},\{\phi'',\phi'\},y'',y';T)\\
&\quad=\sum_{\{L\}}\Psi_{\{L\}}^{\mathrm{SU}(n-1)*}(\{\theta',\phi'\})\Psi_{\{L\}}^{\mathrm{SU}(n-1)}(\{\theta'',\phi''\})K_{k_1,L}^{S_2}(r'',r',y'',y';T)
\end{aligned}
\tag{4.30}
$$

with the kernel $K_{k_1,L}^{S_2}(T)$ given by

$$
\begin{aligned}
&K_{k_1,L}^{S_2}(r'',r',y'',y';T)\\
&\quad=(y'y'')^n(r'r'')^{1-n}\int\frac{\mathcal{D}y(t)}{y^2}\int\mathcal{D}r(t)\\
&\qquad\times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\frac{\dot y^2+\dot r^2}{y^2}-\frac{\hbar^2}{2m}y^2\left(k_1^2r^2+\frac{(L+n-2)^2-\frac14}{r^2}\right.\right.\right.\\
&\qquad\left.\left.\left.-2k_1\sum_{k=2}^{n}l_k+k_1^2y^2\right)\right]\mathrm{d}t\right\}\\
&\quad=(y'y'')^{n-1/2}(r'r'')^{(3-2n)/2}\sum_{N=0}^{\infty}R_N^{L+n-2}(r')R_N^{L+n-2}(r'')\\
&\qquad\times K_{k_1,L,N}(y'',y';T)
\end{aligned}
\tag{4.31}
$$

with the radial wavefunctions (4.18), and $K_{k_1,L,N}(T)$ is completely analogous as in the path integral (4.19) given by ($\omega = \hbar|k_1|/m$)

$$
\begin{aligned}
&K^{S_2}_{k_1,L,N}(y'',y';T)\\
&\quad = \exp\left[-\frac{\mathrm{i}\hbar T}{8m}(4n^2-1)\right]\int\frac{\mathcal{D}y(t)}{y}\\
&\qquad \times\exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\frac{\dot y^2}{y^2}-\frac{\hbar^2k_1^2}{2m}y^4-E_\lambda y^2\right]\mathrm{d}t\right\}.\\
&\quad = \frac{\hbar}{2\pi^2m\omega\sqrt{y'y''}}\int_0^\infty \mathrm{d}p\,p\sinh\pi p\\
&\qquad \times\left|\Gamma\left[\frac{1}{2}\left(1+\mathrm{i}p+\frac{E_\lambda}{\hbar\omega}\right)\right]\right|^2\exp\left[-\frac{\mathrm{i}\hbar T}{2m}(p^2+n^2)\right]\\
&\qquad \times W_{-E_\lambda/2\hbar\omega,\mathrm{i}p/2}\left(\frac{m\omega}{\hbar}y''^2\right)W_{-E_\lambda/2\hbar\omega,\mathrm{i}p/2}\left(\frac{m\omega}{\hbar}y'^2\right)
\end{aligned}\tag{4.32}
$$

with the quantity E_λ

$$
E_\lambda = \frac{\hbar^2}{m}\left[|k_1|(2N+L+n-1)-k_1\sum_{k=2}^{n}l_k\right].\tag{4.33}
$$

Note that $E_\lambda > 0$, due to the construction of the quantity L, and again only a continuous spectrum is allowed [32]. Therefore we have completed an eigenfunction expansion of the path integral on S_2 in terms of $\mathrm{SU}(n-1)$ coordinates as follows

$$
\begin{aligned}
&K^{S_2}(\{x'',y''\}_{k=2}^n,\{x',y'\}_{k=2}^n,x_1'',x_1',y'',y';T)\\
&\quad \equiv K^{S_2}(\{\theta'',\theta'\}_{k=2}^n,\{\phi'',\phi'\}_{k=2}^n,x_1'',x_1',y'',y';T)\\
&\quad = \int_{-\infty}^{\infty}\mathrm{d}k_1\sum_{\{L\}}\sum_{N=0}^{\infty}\int_0^\infty \mathrm{d}p\,\Psi^{S_2\,*}_{k_1,\{L\},N,p}(x_1',\{\theta',\phi'\},r',y')\\
&\qquad \times\Psi^{S_2}_{k_1,\{L\},N,p}(x_1'',\{\theta'',\phi''\},r'',y'')\mathrm{e}^{-\mathrm{i}TE_p/\hbar}
\end{aligned}\tag{4.34}
$$

with the wavefunctions

$$
\Psi^{S_2}_{k_1,\{L\},N,p}(x_1,\{\theta,\phi\},r,y) = \frac{\mathrm{e}^{\mathrm{i}k_1x_1}}{\sqrt{2\pi}}\Psi^{\mathrm{SU}(n-1)}_{\{L\}}(\{\theta,\phi\})r^{(3-2n)/2}R_N^{L+n-2}(r)\Phi(p)\tag{4.35}
$$

with $\Phi_p(y)$ as in equation (4.25) and the energy spectrum (4.23).

5. Quantum motion on the hyperbolic spaces of rank one in pseudospherical polar coordinates

There is one further hyperbolic space generalizing the Poincaré upper half-plane based on an hyperboloid $Q_{-1}^{(n,1)}$ on quaternions [16, 51, 52]. Following [16] the metric in the space $S_3 \cong \mathrm{Sp}(n,1)\,/[\mathrm{Sp}(1)\times\mathrm{Sp}(n)]$ is given by

$$\begin{aligned} \mathrm{d}s^2 = \frac{\mathrm{d}y^2}{y^2} + \frac{1}{y^2}\sum_{k=2}^{n}(\mathrm{d}z_k\mathrm{d}z_k^* + \mathrm{d}z_{n+k}\mathrm{d}z_{k+n}^*) \\ + \frac{1}{y^4}\left(\mathrm{d}x_1 + \mathrm{Im}\sum_{k=2}^{n}(z_k^*\mathrm{d}z_k + z_{n+k}^*\mathrm{d}z_{k+n})\right)^2 \\ + \frac{1}{y^4}\left|\mathrm{d}z_{n+1} + \sum_{k=2}^{n}(z_{n+k}\mathrm{d}z_k - z_k\mathrm{d}z_{k+n})\right|^2 . \end{aligned} \tag{5.1}$$

Here $y > 0$, $x_1 \in \mathbb{R}$, and $z_k = x_k + \mathrm{i}y_k \in \mathbb{C}$ $(k = 2,\ldots,2n)$. However, there seems to be no obvious solution in these particular coordinates similar to the two previous cases. But we can use some general results of harmonic analysis on hyperbolic spaces to set up a path integral formulation. We consider a hyperbolic space X as a quotient space of a Gelfand pair (G,K), $X = G/K$. This property allows an Iwasawa decomposition according to the direct sum of the algebra on G, $\mathcal{G}$, such that $\mathcal{G} = \mathcal{K}+\mathcal{A}+\mathcal{N}$ [16, 52–55]. The root system of the pair $(\mathcal{G},\mathcal{A})$ is denoted by P. The Laplace–Beltrami operator (Casimir operator) on X then is the invariant operator on X with respect to the group actions. The root system in these spaces can be taken with all roots positive (restricted set P^+) and decomposed into two systems, α and 2α, such that if $\mu \in P_{2\alpha}$ then $\mu/2$ is not a root. The subspaces $\mathcal{G}(\alpha)$ and $\mathcal{G}(2\alpha)$ have the dimensions m_α and $m_{2\alpha}$, respectively. Furthermore, the algebra $\mathcal{G}$ can be written as a Cartan decomposition $\mathcal{G} = \mathcal{K} + \mathcal{P}$, $\mathcal{P}$ being the orthogonal complement of $\mathcal{K}$ (note $\mathcal{A} \subset \mathcal{P}$). In the cases in question the subspace corresponding to the algebra $\mathcal{P}^+$ can be represented as a $(m_\alpha + m_{2\alpha} + 1)$-dimensional sphere, denoted as $S^{(m_\alpha+m_{2\alpha})}$.

Following [53, 54] the Laplace–Beltrami operator in terms of pseudopolar coordinates on a hyperbolic symmetric space of rank one can be rewritten as

$$\begin{aligned} \Delta_{\mathrm{LB}}^{G/K} = \frac{\partial^2}{\partial\tau^2} + (m_\alpha\coth\tau + 2m_{2\alpha}\coth 2\tau)\frac{\partial}{\partial\tau} \\ - \left[\frac{\mathcal{L}^{(\mu^+)}}{\sinh^2\tau} + \left(\frac{1}{\sinh^2 2\tau} - \frac{1}{\sinh^2\tau}\right)\mathcal{L}^{(2\mu)}\right] . \end{aligned} \tag{5.2}$$

The operators $\mathcal{L}^{(\mu^+)}$ and $\mathcal{L}^{(2\mu)}$ act on the space of the root systems $\mathcal{G}(\alpha^+)$ (all positive roots) and $\mathcal{G}(2\alpha)$, respectively. Now pick a representation, corresponding to the algebra $\mathcal{P}^+$, i.e. a solution of the Laplace–Beltrami operator on the sphere [illegible]

$$\Psi_{k_1,\{L\},N,p}^{S_2}(x_1,\{\theta,\phi\},r,y) = \frac{\mathrm{e}^{\mathrm{i}k_1x_1}}{\sqrt{2\pi}}\Psi_{\{L\}}^{\mathrm{SU}(n-1)}(\{\theta,\phi\})r^{(3-2n)/2}R_N^{L+n-2}(r)\Phi(p) \tag{4.35}$$

with $\Phi_p(y)$ as in equation (4.25) and the energy spectrum (4.23).

Now introducing in the usual way momentum operators p_τ according to

$$p_\tau = \frac{\hbar}{\mathrm{i}}\left(\frac{\partial}{\partial\tau} + \frac{\Gamma_\tau}{2}\right) \qquad \Gamma_\tau = m_\alpha \coth\tau + m_{2\alpha}\coth 2\tau \tag{5.4}$$

we obtain the separated Hamiltonian corresponding to the hyperbolic coordinate τ as follows

$$H^{G/K} = \frac{1}{2m}p_\tau^2 + \frac{\hbar^2}{8m}\left[\frac{(2l + m_\alpha + m_{2\alpha} - 1)^2 - 1}{\sinh^2\tau} + \frac{2(l + m_{2\alpha} - 1)^2 - 1}{\cosh^2\tau}\right] + E_0^{G/K} \tag{5.5}$$

with the zero-point energy given by

$$E_0^{G/K} = \frac{\hbar^2}{8m}(m_\alpha + 2m_{2\alpha})^2. \tag{5.6}$$

Therefore we obtain for the separated path integral on the hyperbolic space $X = G/K$

$$\begin{aligned} &K^{G/K}(\Omega^{(\alpha^+)''}, \Omega^{(\alpha^+)'}, \Omega^{(2\alpha)''}, \Omega^{(2\alpha)'}, \tau'', \tau'; T) \\ &\quad = \sum_l S_l(\Omega^{(\alpha^+)''}) S_l^*(\Omega^{(\alpha^+)'}) S_l(\Omega^{(2\alpha)''}) S_l^*(\Omega^{(2\alpha)'}) K_l^{G/K}(\tau'', \tau'; T) \end{aligned} \tag{5.7}$$

with spherical harmonics on the subspaces $\mathcal{G}(\alpha^+)$ and $\mathcal{G}(2\alpha)$, respectively, and the kernel $K_l(T)$ given by (cf appendix B) a path integral for a modified Pöschl–Teller potential

$$\begin{aligned} &K_l^{G/K}(\tau'', \tau'; T) \\ &\quad = \mathrm{e}^{-\mathrm{i}E_0^{G/K}T/\hbar} \int \mathcal{D}\tau(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left[\frac{m}{2}\dot{\tau}^2 - \frac{\hbar^2}{8m}\right.\right. \\ &\qquad \left.\left. \times \left(\frac{(2l + m_\alpha + m_{2\alpha} - 1)^2 - 1}{\sinh^2\tau} + \frac{(2l + m_{2\alpha} - 1)^2 - 1}{\cosh^2\tau}\right)\right] \mathrm{d}t\right\} \\ &\quad = \int_0^\infty \mathrm{d}p\, \mathrm{e}^{-\mathrm{i}E_p^{G/K}/\hbar} \Psi_{p,l}^{G/K}(\tau'') \Psi_{p,l}^{G/K\,*}(\tau'). \end{aligned} \tag{5.8}$$

The energy spectrum has the form

$$E_p^{G/K} = \frac{\hbar^2}{2m}\left[p^2 + \frac{(m_\alpha + 2m_{2\alpha})^2}{4}\right] \tag{5.9}$$

and the wavefunctions are given by

$$\begin{aligned} \Psi_{p,l}^{G/K}(\tau) = N_p^{G/K} &(\tanh\tau)^{l + \frac{1}{2}(m_\alpha + m_{2\alpha})} (\cosh\tau)^{\mathrm{i}p} \\ &\times {}_2F_1[l + \tfrac{1}{2}(\tfrac{1}{2}m_\alpha + m_{2\alpha} - \mathrm{i}p), \tfrac{1}{2}(\tfrac{1}{2}m_\alpha + 1 - \mathrm{i}p); l + \tfrac{1}{2}(m_\alpha + m_{2\alpha} + 1); \\ &\tanh^2\tau] \end{aligned} \tag{5.10a}$$

$$\begin{aligned} N_p^{G/K} = \frac{\sqrt{p\sinh\pi p/2\pi^2}}{\Gamma[l + \frac{1}{2}(m_\alpha + m_{2\alpha} + 1)]} &\Gamma[l + \tfrac{1}{2}(\tfrac{1}{2}m_\alpha + m_{2\alpha} + \mathrm{i}p)] \\ &\times \Gamma[\tfrac{1}{2}(\tfrac{1}{2}m_\alpha + 1 + \mathrm{i}p)]. \end{aligned} \tag{5.10b}$$

The energy spectrum (5.9) including the zero-point energy (5.6) E_0 is valid for all hyperbolic spaces $X = G/K$ of rank one. We now specialize to the four cases of hyperbolic spaces of rank one.

(i) The space S_1. In the case of S_1 we have $m_\alpha = n-1$ and $m_{2\alpha} = 0$. We find E_p as in equation (3.21) ($D = n-1$) and the radial wavefunctions are the radial wavefunctions on the pseudosphere $\Lambda^{(n)}$ [41]

$$\Psi^{S_1}_{p,l}(\tau) = \frac{\Gamma(\mathrm{i}p + l + (n-1)/2)}{\Gamma(\mathrm{i}p)}(\sinh\tau)^{(2-n)/2}\mathcal{P}^{(2-n)/2}_{\mathrm{i}p-1/2}(\cosh\tau). \tag{5.11}$$

(ii) The space S_2. In the case of S_2 we have $m_\alpha = 2(n-1)$, $m_{2\alpha} = 1$, we find for E_p the value of equation (4.23) and the radial wavefunctions have the form

$$\Psi^{S_2}_{p,l}(\tau) = N^{S_2}_p(\tanh\tau)^{n+l-1/2}(\cosh\tau)^{\mathrm{i}p}{}_2F_1\left(l + \frac{n-\mathrm{i}p}{2}, \frac{n-\mathrm{i}p}{2}; n+l; \tanh^2\tau\right) \tag{5.12a}$$

$$N^{S_2}_p = \frac{1}{\Gamma(n+l)}\sqrt{\frac{p\sinh\pi p}{2\pi^2}}\,\Gamma\left(l + \frac{n+\mathrm{i}p}{2}\right)\Gamma\left(\frac{n+\mathrm{i}p}{2}\right). \tag{5.12b}$$

(iii) The space S_3. Having achieved the result for the general case we can now also address the case of the space S_3, where $m_\alpha = 4(n-1)$ and $m_{2\alpha} = 3$. Hence

$$E^{S_3}_p = \frac{\hbar^2}{2m}[p^2 + (2n+1)^2] \tag{5.13}$$

with the zero-point energy given by

$$E^{S_3}_0 = \frac{\hbar^2}{2m}(2n+1)^2. \tag{5.14}$$

The radial wavefunctions have the form

$$\Psi^{S_3}_{p,l}(\tau) = N^{S_3}_p(\tanh\tau)^{2n+l-1/2}(\cosh\tau)^{\mathrm{i}p} \times {}_2F_1\left(l + n + \frac{1-\mathrm{i}p}{2}, n - \frac{1+\mathrm{i}p}{2}; 2n+l; \tanh^2\tau\right) \tag{5.15a}$$

$$N^{S_3}_p = \frac{1}{\Gamma(2n+l)}\sqrt{\frac{p\sinh\pi p}{2\pi^2}}\,\Gamma\left(l + n + \frac{1+\mathrm{i}p}{2}\right)\Gamma\left(n - \frac{1+\mathrm{i}p}{2}\right). \tag{5.15b}$$

(iv) The exceptional case. We can even manage the exceptional space $S_4 \cong F_{4(-20)}/\mathrm{Spin}(9)$ [21, 24]. Here $m_\alpha = 8$ and $m_{2\alpha} = 7$. Consequently we have

$$E^{S_4}_p = \frac{\hbar^2}{2m}(p^2 + 11^2) \tag{5.16}$$

with the zero-point energy

$$E^{S_4}_0 = 121\frac{\hbar^2}{2m}. \tag{5.17}$$

The radial wavefunctions are given by

$$\Psi_{p,l}^{S_4}(\tau) = N_p^{S_4}(\tanh\tau)^{l+8-1/2}(\cosh\tau)^{\mathrm{i}p}{}_2F_1\left(l+\frac{11-\mathrm{i}p}{2},\frac{5-\mathrm{i}p}{2};l+8;\tanh^2\tau\right) \tag{5.18a}$$

$$N_p^{S_4} = \frac{1}{\Gamma(l+8)}\sqrt{\frac{p\sinh\pi p}{2\pi^2}}\,\Gamma\left(l+5+\frac{1+\mathrm{i}p}{2}\right)\Gamma\left(2+\frac{1+\mathrm{i}p}{2}\right). \tag{5.18b}$$

6. Summary

We have discussed path integration of the quantum motion on hyperbolic spaces of rank one. For the space S_1 we have calculated the Green function explicitly

$$G^{\mathcal{H}^n}(\{x'',x'\},y'',y';E) = \frac{m}{\pi\hbar}\left(-\frac{1}{2\pi\sinh d}\right)^{(D-3)/2} \mathcal{Q}^{(D-3)/2}_{-1/2-\mathrm{i}\sqrt{2m(E-E_0)}/\hbar}(\cosh d). \tag{6.1}$$

The energy spectrum was given by

$$E_p^{\mathcal{H}^n} = \frac{\hbar^2}{2m}\left[p^2+\frac{(D-2)^2}{4}\right] \tag{6.2}$$

with the largest lower bound

$$E_0^{\mathcal{H}^n} = \frac{\hbar^2(D-2)^2}{8m}. \tag{6.3}$$

Only a continuous spectrum occurred. We have evaluated the wavefunctions and the energy spectrum in rectangular and $(D-2)$-dimensional polar coordinates, respectively. In the case of the incorporation of magnetic fields (3.34) we obtained continuous states with spectrum

$$E_p^{\mathcal{H}^n,b} = \frac{\hbar^2}{2m}\left[p^2+b^2+\frac{(D-2)^2}{4}\right] \tag{6.4}$$

as well as bound states with energy levels

$$E_n^{\mathcal{H}^n,b} = -\frac{\hbar^2}{8m}(2\alpha|k|-2n-1)^2+\frac{\hbar^2}{2m}\left[b^2+\frac{(D-2)^2}{4}\right]$$
$$\alpha = \frac{b\cdot k}{|k|^2} \qquad n=0,\ldots,N_M<\alpha-\tfrac{1}{2}. \tag{6.5}$$

In the space S_2 we determined the spectrum and the wavefunctions in rectangular and $\mathrm{SU}(n-1)$ polar coordinates, respectively, as well.

In all spaces the spectrum and the wavefunctions were determined in an appropriate pseudopolar coordinate system according to the general theory of hyperbolic spaces. All cases could therefore be treated simultaneously and we obtained

$$\begin{aligned} K^{G/K}&(\Omega^{(\alpha^+)''},\Omega^{(\alpha^+)'},\Omega^{(2\alpha)''},\Omega^{(2\alpha)'},\tau'',\tau';T) \\ &= \sum_l S_l(\Omega^{(\alpha^+)''})S_l^*(\Omega^{(\alpha^+)'})S_l(\Omega^{(2\alpha)''})S_l^*(\Omega^{(2\alpha)'}) \\ &\quad \times \int_0^\infty \mathrm{d}p\, \mathrm{e}^{-\mathrm{i}E_p^{G/K}/\hbar}\Psi_{p,l}^{G/K}(\tau'')\Psi_{p,l}^{G/K\,*}(\tau') \end{aligned} \tag{6.6}$$

with the energy spectrum

$$E_p^{G/K} = \frac{\hbar^2 p^2}{2m} + E_0^{G/K} \qquad E_0^{G/K} = \frac{\hbar^2}{8m}(m_\alpha + 2m_{2\alpha})^2 \tag{6.7}$$

and the wavefunctions (5.10). Thus we have completed exact path integration in further Riemannian spaces with constant negative curvature.

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Appendix A: evaluation in SU$(n-1)$ polar coordinates

We want to evaluate the path integral (4.29) by separating the SU$(n-1)$ coordinates. Let us look at the path integral

$$\begin{aligned} K_{k_1}^{\mathrm{SU}(n-1)}&(r'',r',\{\theta'',\theta'\},\{\phi'',\phi'\};T) \\ &= \int r^{2n-3}\mathcal{D}r(t)\int\prod_{k=2}^{n-1}\cos\theta_k(\sin\theta_k)^{2k-3}\mathcal{D}\theta_k(t)\int\prod_{j=2}^{n}\mathcal{D}\phi_j(t) \\ &\quad\times\exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''}\left(\frac{m}{2}\left\{\dot r^2 + r^2\left[\dot\theta_{n-1}^2 + \cos^2\theta_{n-1}\dot\phi_n^2 + \cdots\right.\right.\right.\right. \\ &\quad\left.\left. + \sin^2\theta_3\left(\dot\theta_3^2 + \cos^2\theta_3\dot\phi_3^2 + \sin^2\theta_2\dot\phi_2^2\right)\ldots\right]\right\} \\ &\quad + \hbar k_1 r^2\left\{\dot\phi_n\cos^2\theta_{n-1} + \sin^2\theta_{n-1}\left[\dot\phi_{n-1}\sin^2\theta_{n-2} + \cdots\right.\right. \\ &\quad\left.\left. + \sin^2\theta_3\left(\dot\phi_3\cos^2\theta_2 + \dot\phi_2\sin^2\theta_2\right)\ldots\right]\right\} \\ &\quad + \frac{\hbar^2}{8mr^2}\left[1 + \frac{1}{\cos^2\theta_{n-1}} + \cdots\right. \\ &\quad\left.\left.\left. + \frac{1}{\sin^2\theta_3}\left(1 + \frac{1}{\cos^2\theta_2} + \frac{1}{\sin^2\theta_2}\right)\ldots\right]\right)\mathrm{d}t\right]. \end{aligned} \tag{A.1}$$

Performing the Fourier expansions for the ϕ_j ($j = 2, \ldots, n$) yields (compare the similar procedure in [50])

$$K_{k_1}^{\mathrm{SU}(n-1)}(r'', r', \{\theta'', \theta'\}, \{\phi'', \phi'\}; T) = \frac{1}{(2\pi)^{n-1}} \prod_{j=2}^{n} \sum_{l_j=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}l_j(\phi_j''-\phi_j')} K_{k_1,\{l\}}^{\mathrm{SU}(n-1)}(r'', r', \{\theta'', \theta'\}; T). \tag{A.2}$$

This gives for the kernel $K_{k_1,\{l\}}^{\mathrm{SU}(n-1)}(T)$

$$\begin{aligned} K_{k_1,\{l\}}^{\mathrm{SU}(n-1)}&(r'', r', \{\theta'', \theta'\}; T) \\ &= \left[\prod_{j=2}^{n-1} \cos\theta_j' \cos\theta_j'' (\sin\theta_j' \sin\theta_j'')^j\right]^{-1/2} \\ &\quad\times \int \prod_{j=1}^{n-1} (\sin\theta_j)^{j-1} \mathcal{D}\theta_j(t) \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \left(\frac{m}{2}\dot{r}^2 + \frac{m}{2} r^2 \{\dot\theta_{n-1}^2\right.\right. \\ &\quad + \sin^2\theta_{n-1}[\dot\theta_{n-2}^2 + \cdots + \sin^2\theta_2 \dot\theta_1^2]\} \\ &\quad - \frac{\hbar^2 k_1^2}{2m} r^2 + \frac{\hbar k_1}{m} \sum_{k=2}^{n} l_k - \frac{\hbar^2}{2mr^2}\left\{\frac{l_n^2 - \frac14}{\cos^2\theta_{n-1}} + \frac{1}{\sin^2\theta_{n-1}}\right. \\ &\quad \left.\left.\left.\times \left[\frac{l_{n-1}^2 - \frac14}{\cos^2\theta_{n-2}} + \cdots + \frac{1}{\sin^2\theta_3}\left(\frac{l_3^2 - \frac14}{\cos^2\theta_2} + \frac{l_2^2 - \frac14}{\sin^2\theta_2}\right) \cdots - \frac14\right]\right\}\right) \mathrm{d}t\right]. \end{aligned} \tag{A.3}$$

The $\{\theta\}$ path integrations are now interrelated Pösch–Teller potential path integrations. The path integral solution of the Pöschl–Teller potential

$$V^{(\lambda,\kappa)}(x) = \frac{\hbar^2}{2m}\left(\frac{\kappa^2 - \frac14}{\cos^2 x} + \frac{\lambda^2 - \frac14}{\sin^2 x}\right) \qquad 0 < x < \tfrac12\pi \tag{A.4}$$

was given by Böhm and Junker [57] and Duru [56] and has the form

$$\begin{aligned} K^{(\mathrm{PT})}(x'', x'; T) &= \int \mathcal{D}x(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \left[\frac{m}{2}\dot{x}^2 - \frac{\hbar^2}{2m}\left(\frac{\kappa^2 - \frac14}{\cos^2 x} + \frac{\lambda^2 - \frac14}{\sin^2 x}\right)\right] \mathrm{d}t\right\} \\ &= \sum_{n=0}^{\infty} \exp\left[-\frac{\mathrm{i}\hbar T}{2m}(\kappa + \lambda + 2n + 1)^2\right] \Psi_n^{(\lambda,\kappa)*}(x') \Psi_n^{(\lambda,\kappa)}(x''). \end{aligned} \tag{A.5}$$

The wavefunctions $\Psi_n^{(\lambda,\kappa)}$ are given by

$$\begin{aligned} \Psi_n^{(\lambda,\kappa)}(x) &= \left[2(\kappa + \lambda + 2n + 1) \frac{n!\Gamma(\kappa + \lambda + n + 1)}{\Gamma(\kappa + n + 1)\Gamma(\lambda + n + 1)}\right]^{1/2} \\ &\quad\times (\sin x)^{\lambda + \frac12} (\cos x)^{\kappa + \frac12} P_n^{(\lambda,\kappa)}(\cos 2x). \end{aligned} \tag{A.6}$$

The $P_n^{(\lambda,\kappa)}$ denote Jacobi polynomials. Proceeding now as in [50] we introduce the quantum numbers

$$\begin{aligned} L_2 &= 2n_2 + |l_2| + |l_3| \\ L_3 &= 2n_3 + L_2 + |l_4| \\ &\vdots \\ L_i &= 2n_i + L_{i-1} + |k_{i+1}| \qquad i = 2, \ldots, n-2 \\ L \equiv L_{n-1} &= 2n_{n-1} + L_{n-3} + |l_{n-1}|. \end{aligned} \tag{A.7}$$

Performing the successive Pöschl–Teller separations we finally obtain

$$\begin{aligned} K_{k_1}^{SU(n-1}(r'', r', \{\theta'', \theta'\}, \{\phi'', \phi'\}; T) &= (r'r'')^{(3-2n)/2} \sum_{\{L\}} \Psi_{\{L\}}^{SU(n-1)*}(\{\theta', \phi'\}) \\ &\quad \times \Psi_{\{L\}}^{SU(n-1)}(\{\theta'', \phi''\}) K_L(r'', r'; T) \end{aligned} \tag{A.8}$$

where $\{L\}$ denotes the summation over all quantum numbers, with the radial kernel

$$\begin{aligned} K_L(r'', r'; T) &= \exp\left(\frac{\mathrm{i}\hbar k_1 T}{m}\right) \int \mathcal{D}r(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \left[\frac{m}{2}\dot{r}^2 - \frac{\hbar^2 k_1^2}{2m} r^2 \right.\right. \\ &\quad \left.\left. - \hbar^2 \frac{(L+n-2)^2 - \frac{1}{4}}{2mr^2}\right] \mathrm{d}t\right\} \end{aligned} \tag{A.9}$$

and the $\Psi_{\{L\}}^{SU(n-1)}$ wavefunctions are given by

$$\begin{aligned} \Psi_{\{L\}}^{SU(n-1)}(\{\theta\}, \{\phi\}) &= \left[\prod_{j=2}^{n-1} \cos\theta_j (\sin\theta_j)^{2j-3}\right]^{-\frac{1}{2}} (2\pi)^{-(n-1)/2} \exp\left(\mathrm{i}\sum_{j=2}^{n} k_i \phi_i\right) \\ &\quad \times \Psi_{n_2}^{(k_2,k_3)}(\theta_2) \ldots \Psi_{n_{n-1}}^{(L_{n-3}+n-3,k_n)}(\theta_{n-1}). \end{aligned} \tag{A.10}$$

Hence the separation of the $\{\theta, \phi\}$ angular $SU(n-1)$ variables is achieved. Note that we also have [58]

$$\begin{aligned} &\prod_{k=2}^{n} \int \mathcal{D}x_k(t) \int \mathcal{D}y_k(t) \exp\left\{\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \left[\frac{m}{2}\sum_{k=2}^{n}(\dot{x}_k^2 + \dot{y}_k^2) \right.\right. \\ &\qquad \left.\left. + \hbar k_1 \sum_{k=2}^{n}(x_k \dot{y}_k - y_k \dot{x}_k)\right] \mathrm{d}t\right\} \\ &\quad = \left(\frac{k_1}{2\pi\mathrm{i}\sin(\hbar k_1 T/m)}\right)^{n-1} \prod_{k=2}^{n} \exp\left\{-\frac{m}{2\mathrm{i}\hbar}\left[\frac{\hbar k_1}{m}\cot\left(\frac{\hbar k_1}{m}T\right) \right.\right. \\ &\qquad \left.\left. \times \left[(x_k'' - x_k')^2 + (y_k'' - y_k')^2\right] + \frac{2\hbar k_1}{m}(x_k' y_k'' - x_k'' y_k')\right]\right\}. \end{aligned} \tag{A.11}$$

Appendix B: path integration on generalized hyperboloids

Let us briefly discuss some general feature of the quantum motion on hyperboloids $Q_{-1}^{(n,1)}$,

$$Q_{-1}^{(n,1)} = |f_1|^2 + |f_2|^2 + \ldots + |f_n| - |f_{n+1}|^2 = -1 \tag{B.1}$$

and the numbers f are $f = 1, e^{i\phi_i}, q$, corresponding to real numbers, complex numbers and quaternions with unit absolute values (a suitable polar decomposition of quaternions are, for example, $q = (\cos\psi_1, i, \sin\psi_1\cos\psi_2, j\sin\psi_1 \sin\psi_2\cos\psi_3, k\sin\psi_1\sin\psi_2\sin\psi_3$ with $(0 \leqslant \psi_3 \leqslant 2\pi,\ 0 \leqslant \psi_{1,2} \leqslant \pi$, with $i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ ik = -ki = j$ and $jk = -kj = i$). Now consider a pseudo-polar coordinate system

$$\begin{aligned}
f_{n+1} &= q_{n+1}\cosh\tau \\
f_n &= q_n \sinh\tau\cos\theta_{n-1} \\
f_{n-1} &= q_{n-1}\sinh\tau\sin\theta_{n-1}\cos\theta_{n-2} \\
&\vdots \\
f_2 &= q_2\sinh\tau\ldots\sin\theta_2\cos\theta_1 \\
f_1 &= q_1\sinh\tau\ldots\sin\theta_2\sin\theta_1
\end{aligned} \tag{B.2}$$

$(0 \leqslant \theta_j \leqslant \frac{1}{2}\pi; j = 1, \ldots, n-1; \tau > 0)$. We attach to this $(n+1)$-dimensional pseudo-polar coordinate system, the numbers $q_i = 1$ $(j = 1, \ldots, n+1)$, complex coordinates $q_i = e^{-\phi_i}$ $(i = 1, \ldots, n+1)$, or quaternion coordinates q_l $(l = 1, \ldots, n+1)$, respectively. That is we consider the free motion on the unit spheres of the group manifolds $SO(n,1)$, $SU(n,1)$ and $Sp(n,1)$, respectively. For $q_l = 1$ $(l = 1, \ldots, n+1)$ we obtain the known result for the pseudosphere $\Lambda^{D-1} \cong S_1$ with a pure continuous spectrum

$$E_p = \frac{\hbar^2}{2m}\left[p^2 + \frac{(n-1)^2}{4}\right] \qquad p > 0. \tag{B.3}$$

For $q_l = e^{-\phi_l}$ $(l = 1, \ldots, n+1)$ and the covering unit sphere of $SU(n,1)$ [50] we get the result for the path integration for 'pure $SU(n,1)$' with a continuous and a discrete spectrum [50], the continuous spectrum having the form

$$E_p = \frac{\hbar^2}{2m}(p^2 + n^2) \qquad p > 0. \tag{B.4}$$

In the next step we take for the q_i quaternions, which means that in a separation procedure for the path integral covering the unit sphere of $Sp(n,1)$, for each set of coordinates $\psi_{l,i}$ $(l = 1, \ldots, n+1, i = 1,2,3)$ we separate a Feynman kernel on the $S^{(3)}$ sphere instead for the case of a circle. The general feature of the $SU(u,v)$ path integration therefore is not changed at all (in appendix A just replace $\theta_{n-1} \to i\tau$ and ϕ_l by the set $\psi_{l,i}$ $(l = 1, \ldots, n+1,\ i = 1,2,3)$, and we obtain essentially the same result (including some slight modification of some quantum numbers, in particular the

l_j arising from the $\mathrm{U}(1)$ integration are now changed according to $l_j \to L_j + 2$ with the L_j the principal quantum numbers of the Laplacian on the $S^{(3)}$ sphere). This procedure gives eventually a path integral for the modified Pöschl–Teller potential

$$V^{(\eta,\nu)}(\tau) = \frac{\hbar^2}{2m}\left(\frac{\eta^2 - \frac{1}{4}}{\sinh^2\tau} - \frac{\nu^2 - \frac{1}{4}}{\cosh^2\tau}\right). \tag{B.5}$$

The path integral solution which can be achieved by means of the $\mathrm{SU}(1,1)$ path integral [57, 59] has the form

$$\begin{aligned} K^{(\mathrm{MPT})}(\tau'', \tau'; T) \\ &= \sum_{n=0}^{N_{\mathrm{M}}} \Phi_n^{(\eta,\nu)*}(\tau')\Phi_n^{(\eta,\nu)}(\tau'')\exp\left\{-\frac{\mathrm{i}\hbar T}{2m}[2(k_1 - k_2 - n) - 1]^2\right\} \\ &\quad + \int_0^\infty \mathrm{d}p\, \Phi_p^{(\eta,\nu)*}(\tau')\Phi_p^{(\eta,\nu)}(\tau'')\exp\left(-\frac{\mathrm{i}\hbar T}{2m}p^2\right). \end{aligned} \tag{B.6}$$

Here $k_1 = \frac{1}{2}(1 \pm \nu)$, $k_2 = \frac{1}{2}(1 \pm \eta)$, and N_{M} denotes the maximal number of bound states with $0, 1, \ldots, N_{\mathrm{M}} < k_1 - k_2 - \frac{1}{2}$. The bound state wavefunctions have the form

$$\begin{aligned} \Phi_n^{(\eta,\nu)} &= N_n^{(\eta,\nu)}(\sinh\tau)^{2k_2-1/2}(\cosh\tau)^{-2k_1+3/2} \\ &\qquad \times {}_2F_1(-k_1 + k_2 + \kappa, -k_1 + k_2 - \kappa + 1; 2k_2; -\sinh^2\tau) \end{aligned} \tag{B.7a}$$

$$N_n^{(\eta,\nu)} = \frac{1}{\Gamma(2k_2)}\left[\frac{2(2\kappa - 1)\Gamma(k_1 + k_1 - \kappa)\Gamma(k_1 + k_2 + \kappa - 1)}{\Gamma(k_1 - k_2 + \kappa)\Gamma(k_1 - k_2 - \kappa + 1)}\right]^{1/2} \tag{B.7b}$$

($\kappa = k_1 - k_2 - n$), and the scattering states are given by

$$\begin{aligned} \Phi_p^{(\eta,\nu)} &= N_p^{(\eta,\nu)}(\cosh\tau)^{2k_1-1/2}(\sinh\tau)^{2k_2-1/2} \\ &\qquad \times {}_2F_1(k_1 + k_2 - \kappa, k_1 + k_2 + \kappa - 1; 2k_2; -\sinh^2\tau) \end{aligned} \tag{B.8a}$$

$$\begin{aligned} N_p^{(\eta,\nu)} &= \frac{1}{\Gamma(2k_2)}\sqrt{\frac{p\sinh\pi p}{2\pi^2}}[\Gamma(k_1 + k_2 - \kappa)\Gamma(-k_1 + k_2 + \kappa) \\ &\qquad \times \Gamma(k_1 + k_2 + \kappa - 1)\Gamma(-k_1 + k_2 - \kappa + 1)]^{1/2} \end{aligned} \tag{B.8b}$$

where $\kappa = \frac{1}{2}(1 + \mathrm{i}p)$. Therefore we obtain a discrete and a continuous spectrum for the quantum motion on the covering unit sphere of $\mathrm{Sp}(n, 1)$, the continuous spectrum being

$$E_p = \frac{\hbar^2}{2m}[p^2 + (2n + 1)^2] \qquad p > 0. \tag{B.9}$$

Hence, in all cases the continuous spectra of the quantum motion on the covering unit spheres of $\mathrm{SO}(n, 1)$, $\mathrm{SU}(n, 1)$ and $\mathrm{Sp}(n, 1)$, respectively, are identical to the spectra on the hyperbolic spaces S_1, S_2 and S_3, respectively.

Let us stress that, in fact, any coordinate system q_n can be attached to the pseudopolar coordinates f_n for any hyperboloid $Q_e^{(p,q)}$; one just has to work out the details. However, the nice group structure leading to the consideration of hyperbolic spaces of rank one is revealed only for the particular cases discussed here.

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